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Noncommutative Φ^4 theory at two loops

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Abstract: We study perturbative aspects of noncommutative field theories. This work is arranged in two parts. First, we review noncommutative field theories in general and discuss both canonical and path integral quantization methods. In the second part, we consider the particular example of noncommutative Φ^4 theory in four dimensions and work out the corresponding effective action and discuss renormalizability of the theory, up to two loops.

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Contents

1. Generalities **2**
- 1.1 Introduction 2
- 1.2 Noncommutative spaces 4
- 1.3 Properties of the star product 5

2. Noncommutative field theory at classical level **6**
- 2.1 Conjugate momentum and equations of motion 7
- 2.2 Noether theorem 7

3. Canonical quantization of noncommutative theories **11**
- 3.1 Scalar theory 11
- 3.2 Fermionic theories 12
- 3.3 Interactions 12

4. Path integral quantization of noncommutative theories **13**
- 4.1 Measures 13
- 4.2 N-point functions and effective action for noncommutative theories 14

5. The effective action for the noncommutative Φ^4 theory **15**
- 5.1 One loop effective action 16
- 5.2 Diagrammatics 17
- 5.3 Diagrammatic expansion of the effective action 18
- 5.4 The effective action at higher orders 19
- 5.5 Planar and nonplanar diagrams 19

6. Renormalization of noncommutative Φ^4 theory **20**
- 6.1 1-loop renormalization of $\Gamma^{(2)}$ 20
- 6.2 1-loop renormalization of $\Gamma^{(4)}$ 22
- 6.3 $\Gamma^{(2)}$ at two loops 24
- 6.4 $\Gamma^{(4)}$ at two loops 27

7. Conclusions and remarks **34**

A. Functional derivatives in star product formalism **35**

B. Phase factors associated with various diagrams **37**

1. Generalities

1.1 Introduction

In the past two years, a lot of work has been devoted to the study of noncommutative field theories, i.e. field theories on the Moyal plane. The main motivation for these theories arises from string theory: the end points of the open strings trapped on a D-brane with a nonzero NSNS two form B-field background turn out to be noncommuting [1]. Then the noncommutative field theories, in particular noncommutative supersymmetric (Yang-Mills) gauge theories appear as the low energy effective theory of such D-branes [2, 3]. Apart from string theory, noncommutative field theories are very interesting as field theories. In general, when we study a field theory, we should emphasize that it is "well behaved". From this point of view, noncommutative field theories are really challenging because they are nonlocal (they involve arbitrarily high orders of the derivatives), and there is a dimensionful parameter, other than the masses, the noncommutativity parameter θ. The nonlocality may have consequences on the "CPT theorem" as well as the causality. On the other hand the dimensionful parameter θ may ruin the renormalizability of the theory. It was shown in [4, 5] that indeed space-time noncommutativity ($\theta_{0i} \neq 0$) leads to a non-unitary theory, while field theories on noncommutative space are well behaved in this respect. However, we should add that the special case of "light-like" noncommutativity has also been discussed and shown that it leads to unitary quantum theories [5, 6].

Similar to the usual field theories, one can build noncommutative version of scalar, Dirac and vector (gauge) theories. The noncommutative scalar theory with Φ^4 interaction is considered in [7]–[10] and it has been shown that this theory is renormalizable up to two loops. Similarly, one can consider the pure noncommutative gauge theories; in particular noncommutative $\mathrm{U}(N)$ theory has been shown to be renormalizable up to one loop [7], [10]–[12]. Adding fermions to the noncommutative $\mathrm{U}(1)$ has also been studied in [13]–[15]. However in this work we will mostly concentrate on the scalar theory.

Usually in studying the noncommutative field theories, the classical properties of these theories are not addressed. At the classical level, the variational method and the Noether theorem are discussed in detailed. We show that the noncommutative version of the Noether theorem holds, i.e. the conserved current for the *external* symmetries (those which are not related to space-time, like gauge symmetry) is the usual one, but the product of the fields is now replaced by the star-product. For the internal symmetries under which the noncommutative parameter $\theta_{mu\nu}$, is invariant (e.g. the translation symmetry) in general the conserved current is not conserved, i.e. its divergence can be expressed as Moyal brackets. However, we should remind that for both the internal and external symmetries, the conserved *charge* is still conserved, provided that the space-time noncommutativity (θ_{0i}) is zero.

To quantize any given field theory there are two approaches which are usually equivalent [16], the path integral and canonical quantizations. We discuss both cases for the noncommutative scalar theory. Along these ideas, this can be done for all fields, fermionic and vector fields. The basic point is that, despite having different interactions, and because the quadratic part of the action is the same for the commutative and noncommutative cases, the *perturbative* Fock space for the commutative theory and its noncommutative counter-part are the same. In the path integral formulation this is reflected in the fact that the integration measures are the same for the commutative case and its noncommutative version.

We then proceed with the explicit loop calculations of the real noncommutative ϕ^4 theory in four dimensions. At one loop , we present the detailed calculation of both two and four point functions. In general, for any noncommutative field theory, the loop diagrams can be classified in the so-called "planar" and "non-planar" graphs. At one loop level, the planar part of diagrams show the same UV divergence structure as the corresponding commutative theory, while the non-planar pieces are UV finite. So, altogether the counter-terms (responsible for the cancellation of UV infinities) have the same structure, but with different numeric factors, compared to the commutative counter-part. However, surprisingly it turns out that this difference in the numeric factors is not going to destroy the (UV) renormalizability of the noncommutative ϕ^4, Yang-Mills and QED theories. As for the non-planar diagrams, we will show that, although being UV finite, they involve IR divergences. This is what is known as UV/IR mixing, which is a general feature of the noncommutative field theories [10]. This UV/IR mixing can be understood intuitively: Let us suppose x and y coordinates be noncommuting, $[x, y] = i\theta$. From the usual operator algebra one can easily conclude that $\Delta x \Delta y \geq \theta$. So, increasing precision in x direction ($\Delta x \to 0$ or UV limit) is naturally related to the $\Delta y \to \infty$ or the IR limit. Although in [7, 17] some arguments have been presented to remove these new IR divergences, they are not yet well-understood.

In this work, to ensure (and improve) the renormalizability issue of the noncommutative field theories we present the detailed calculations for the noncommutative ϕ^4 theory up to two loops. We show that, as it is expected from renormalizable theory, again (UV) infinities are canceled while we still remain with the IR divergences originating from the non-planar diagrams. The peculiar feature which we find is the fact that the counter-terms accounting for the UV divergences at two loops, like the one loop case, *do not depend on* θ.

Also we note that the noncommutative parameter, θ, will not receive any quantum corrections up to two loops, and we expect this to remain at all loop levels and also even non-perturbatively. From the string theory point of view this is expected; there the B-field (which leads to noncommutativity on the D-brane worldvolume) is one of the moduli of the theory preserved by supersymmetry. We should remind the fact that for the gauge theories on noncommutative torus the "Morita equivalence" (e.g. see ref. [18, 19]) the noncommutativity parameter is defined up to the $SL(2,\mathbb{Z})$ transformation [3, 19, 20], however here wo do not discuss the compact cases.

This work is organized as follows. In section 2 we present some classical aspects of noncommutative theories deriving the equations of motion and the Noether theorem. In section 3 we briefly discuss the canonical quantization procedure for noncommutative theories. In the next section we describe the path integral quantization which we are going to use in section 5 to derive the two loops expression for the effective action of the noncommutative Φ^4 theory. In section 6 we present calculations which proves the (UV) renormalizability of the Φ^4 theory at two loops. The last section is devoted to remarks and conclusions.

1.2 Noncommutative spaces

In the usual quantum mechanics we have the well known commutation relations:

$$\begin{aligned} \left[\hat{X}_i, \hat{P}_j\right] &= i\hbar\delta_{ij} \qquad \text{and} \\ \left[\hat{X}_i, \hat{X}_j\right] &= \left[\hat{P}_i, \hat{P}_j\right] = 0\,. \end{aligned} \tag{1.1}$$

However there is no evidence that at very short distances (or very high energies) these relations should still be true. Then a natural generalization of above is to take the coordinates which do not commute any more,

$$\left[\hat{X}_i, \hat{X}_j\right] = i\theta_{ij}\,, \tag{1.2}$$

where θ_{ij} is a *constant* of dimension $[L]^2$. An immediate remark is that introducing this kind of commutation relation between coordinates the Lorentz invariance is spoiled explicitly. We should remember however that we assumed this feature to appear only at very short distances, i.e. for $\theta \to 0$ we should recover the Lorentz symmetry. This is one of the main constraints on our noncommutative field theories: at least at classical level, in the limit $\theta \to 0$ we should find a previously known commutative field theory.[1] In general (1.2) can be extended to space-time coordinates:

$$\left[\hat{X}_\mu, \hat{X}_\nu\right] = i\theta_{\mu\nu}\,. \tag{1.3}$$

Here after we call a space with the above commutation relations as a noncommutative space.

To construct the perturbative field theory formulation, it is more convenient to use fields which are some functions and not operator valued objects. To pass to such fields while keeping (1.3) property one should redefine the multiplication law of

[1]However this in general does not imply the reverse: the noncommutative extension of a given theory is not unique. As an example SO(2) and U(1) gauge theories are the same, but in noncommutative version they are different [21].

functional (field) space. This new multiplication is induced from (1.3) through the so called Weyl-Moyal correspondence [22]:

$$\hat{\Phi}(\hat{X}) \longleftrightarrow \Phi(x) \ ;$$

$$\begin{aligned}\hat{\Phi}(\hat{X}) &= \int_\alpha e^{i\alpha\hat{X}}\phi(\alpha)d\alpha \\ \phi(\alpha) &= \int e^{-i\alpha x}\Phi(x)dx\,,\end{aligned} \tag{1.4}$$

where α and x are real variables. Then,

$$\begin{aligned}\hat{\Phi}_1(\hat{X})\hat{\Phi}_2(\hat{X}) &= \iint_{\alpha\beta} e^{i\alpha\hat{X}}\phi(\alpha)e^{i\beta\hat{X}}\phi(\beta)d\alpha\, d\beta \\ &= \iint_{\alpha\beta} e^{i(\alpha+\beta)\hat{X}-\frac{1}{2}\alpha_\mu\beta_\nu[\hat{X}_\mu,\hat{X}_\nu]}\phi_1(\alpha)\phi_2(\beta)d\alpha\, d\beta\,,\end{aligned} \tag{1.5}$$

and hence,

$$\hat{\Phi}_1(\hat{X})\hat{\Phi}_2(\hat{X}) \longleftrightarrow \Big(\Phi_1 \star \Phi_2\Big)(x)\,, \tag{1.6}$$

$$\Big(\Phi_1 \star \Phi_2\Big)(x) \equiv \left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}\Phi_1(x+\xi)\Phi_2(x+\eta)\right]_{\xi=\eta=0}. \tag{1.7}$$

This suggests that we can work on a usual commutative space for which the multiplication operation is modified to the so called star product (1.7). It is easy to check that the Moyal bracket (the commutator in which the product is modified with a star product) of two coordinates x_μ and x_ν gives exactly the desired commutation relations, (1.3)

$$[x_\mu, x_\nu]_{\mathrm{MB}} = i\theta_{\mu\nu}\,. \tag{1.8}$$

1.3 Properties of the star product

Here we summarize some useful identities of the star product algebra.

1. The star product between exponentials:

$$\begin{aligned}e^{ikx} \star e^{iqx} &= e^{i(k+q)x}e^{-\frac{i}{2}(k\theta q)}\,, \qquad \text{where} \\ k\theta p &\equiv k^\mu p^\nu \theta_{\mu\nu}\,.\end{aligned} \tag{1.9}$$

2. Momentum space representation:

   let $\tilde{f}(k)$ and $\tilde{g}(k)$ be the Fourier components of f and g. Then using (1.9)

$$\big(f \star g\big)(x) = \int d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q\, \tilde{f}(k)\tilde{g}(q)e^{-\frac{i}{2}(k\theta q)}e^{i(k+q)x}\,. \tag{1.10}$$

3. Associativity:

$$\Big[\big(f \star g\big) \star h\Big](x) = \Big[f \star \big(g \star h\big)\Big](x)\,, \tag{1.11}$$

which can be proved immediately if we go to momentum space.

$$\begin{aligned}
\text{r.h.s.} &= \int đ^4k đ^4q đ^4p \tilde{f}(k)\tilde{g}(q)\tilde{h}(p) e^{-\frac{i}{2}(k\theta q)} e^{-\frac{i}{2}((k+q)\theta p)} e^{i(k+q+p)x}\,, \quad \text{and} \\
\text{l.h.s.} &= \int đ^4k đ^4q đ^4p \tilde{f}(k)\tilde{g}(q)\tilde{h}(p) e^{-\frac{i}{2}(q\theta p)} e^{-\frac{i}{2}(k\theta(q+p))} e^{i(k+q+p)x}\,.
\end{aligned} \tag{1.12}$$

4. Star products under integral sign

$$\int \big(f \star g\big)(x) d^4x = \int \big(g \star f\big)(x) d^4x = \int \big(f \cdot g\big)(x) d^4x\,. \tag{1.13}$$

Using (1.10) we can immediately perform the integration over x which will give a $\delta^4(k+q)$. Due to the antisymmetry of θ the exponent vanishes and so:

$$\begin{aligned}
\int \big(f \star g\big)(x) d^4x &= \int đ^4k \tilde{f}(k)\tilde{g}(-k) \\
&= \int \big(f \cdot g\big)(x) d^4x\,.
\end{aligned} \tag{1.14}$$

From (1.13) we can deduce the cyclic property:

$$\int \big(f_1 \star f_2 \star \cdots \star f_n\big)(x) d^4x = \int \big(f_n \star f_1 \star \cdots \star f_{n-1}\big)(x) d^4x\,. \tag{1.15}$$

5. Complex conjugation.

$$\big(f \star g\big)^* = g^* \star f^*\,. \tag{1.16}$$

It is obvious that if f is a real function then $f \star f$ is also real.

2. Noncommutative field theory at classical level

As we have seen in the previous section the way to treat the noncommutative theories is to modify the usual product of fields with the star product. So, for example, the action for the noncommutative analog of the real Φ^4 theory will be:

$$S[\Phi] = \int d^4x \left[\frac{1}{2}\partial_\mu\Phi \star \partial^\mu\Phi - \frac{m^2}{2}\Phi \star \Phi - \frac{\lambda}{4!}\Phi \star \Phi \star \Phi \star \Phi\right]. \tag{2.1}$$

Thanks to (1.13), the quadratic part of the action is the same as the commutative case. Therefore the only thing which is modified is the interaction. This is a very important point to keep in mind that the free theory is *the same* as in the commutative case. However we should remind that this is not true for the topologically non-trivial spaces [23].

2.1 Conjugate momentum and equations of motion

The classical equations of motion, similar to the commutative case, are obtained by minimizing the action, i.e.

$$\frac{\delta S}{\delta \Phi} = 0 \,. \tag{2.2}$$

The right meaning of the functional derivatives can be found in the appendix A. Using these results we can write the equation of motion for the scalar filed theory with a Φ^4 interaction:

$$(\Box + m^2)\Phi = -\frac{\lambda}{3!}\big(\Phi \star \Phi \star \Phi\big)(x) \,. \tag{2.3}$$

In order to find the conjugate momentum we should first distinguish two major cases:

- $\theta_{0i} = 0$,
- $\theta_{0i} \neq 0$.

$\theta_{0i} = 0$. In this case the only place where we encounter time derivatives is the kinetic term so the conjugate momentum is the same as in the commutative case.

$\theta_{0i} \neq 0$. This case is more delicate since we have infinite number of time derivatives in the interaction term. It is obvious right from the beginning that there is something non-trivial in this case; the conjugate momentum depends on the interaction terms. The infinite number of time derivatives suggests us that the theory is nonlocal in time so causality may be violated [5]. It was also shown that at quantum level unitarity is not preserved any more [4].[2] For these reasons we will restrict ourselves only to the case with $\theta_{0i} = 0$ from now on.

2.2 Noether theorem

Now that we have developed the functional differentiation we can extend the Noether theorem to the noncommutative field theories. Suppose our action has a *global continuous* symmetry. For an infinitesimal transformation we can write:

$$S[\Phi] = S[\Phi + \varepsilon \mathcal{F}(\Phi)] \,, \quad \text{with} \quad \varepsilon = \text{constant} \,. \tag{2.4}$$

Taking now an x-dependent ε we define the current J through the relation:

$$S[\Phi + \varepsilon(x)\mathcal{F}] - S[\Phi] \equiv -\int J^\mu\big(\Phi(x)\big)\partial_\mu \, \varepsilon(x) \,. \tag{2.5}$$

[2]The case of $\theta_{0i} \neq 0$ for a cylinder has recently been discussed in [24].

By definition the action is stationary for *any* field variation around the classical path i.e. $\delta S/\delta\Phi = 0$. In particular for $\delta\Phi = \varepsilon(x)\mathcal{F}$ eq. (2.5) becomes:

$$\int J^\mu\big(\Phi(x)\big)\partial_\mu\,\varepsilon(x)\Big|_{\substack{\text{classical}\\ \text{path}}} = 0\,. \tag{2.6}$$

Integrating by parts we find:

$$\int \partial_\mu J^\mu\big(\Phi(x)\big)\,\varepsilon(x)d^4x = 0\,, \tag{2.7}$$

for any $\varepsilon(x)$. So the current J is conserved. This result is very general and it can be applied for any kind of noncommutative theory. The notion of conserved current is a little different from the commutative case. Due to the property (1.13)

$$\int [f,g]_{\mathrm{MB}}d^4x = 0 \tag{2.8}$$

so the most we can say from eq. (2.7) is:

$$\partial_\mu J^\mu = [f,g]_{\mathrm{MB}}\ , \tag{2.9}$$

for some proper functions f, g. This result is somehow natural since in the limit $\theta \to 0$ the Moyal bracket vanishes and we recover the classical result $\partial_\mu J^\mu = 0$.

Let us see now what happens to the charge which in the commutative case was conserved

$$Q = \int J^0 d^3x\ . \tag{2.10}$$

Since we are considering only the case $\theta_{0i} = 0$, we can repeat the argument we have used to prove (1.13) for the case of integration only over the space coordinates and we conclude

$$\int [f,g]_{\substack{\mathrm{MB}\\ \theta_{0i}=0}} d^3x = 0\,. \tag{2.11}$$

This means that if we integrate (2.9) over the spatial coordinates we get:

$$\partial_0 \int J^0 d^3x + \int \vec{\nabla}\cdot\vec{J}d^3x = 0 \tag{2.12}$$

and from here we can say that, as in the commutative case, the charge Q is conserved. Note that this is true only for $\theta_{0i} = 0$ and for $\theta_{0i} \neq 0$ even the notion of the conserved charge is ill-defined.

For *external* (space-time) symmetries, e.g. translations, under which the noncommutativity parameter, $\theta_{\mu\nu}$, remains invariant, one can also work out the corresponding conserved current. For clarity, let us consider this particular case:

$$\begin{aligned} \Phi &\longrightarrow \Phi + \delta\Phi\,, \\ \delta\Phi &= \varepsilon^\mu\partial_\mu\Phi\,, \\ x_\mu &\longrightarrow x_\mu + \varepsilon_\mu\,. \end{aligned} \tag{2.13}$$

For the action of the form:

$$S=\int d^4x\mathcal{L}(\Phi,\partial\,\Phi)\,, \tag{2.14}$$

where

$$\mathcal{L}=\frac{1}{2}\left(\partial_\mu\Phi\star\partial^\mu\Phi-m^2\Phi\star\Phi\right)+V_\star(\Phi) \tag{2.15}$$

we find:

$$\delta S|_{\delta\Phi=\varepsilon^\mu\partial_\mu\Phi}=\int d^4x\left[\frac{1}{2}\partial_\mu\left(\partial^\mu\Phi\star\partial_\nu\Phi\varepsilon^\nu+\varepsilon^\nu\,\partial_\nu\Phi\star\partial^\mu\Phi\right)-\partial_\mu\left(\varepsilon^\mu\mathcal{L}\right)\right]. \tag{2.16}$$

If we take Φ to be the classical path, i.e. $\delta S=0$ we can write:

$$\int\partial_\mu\left(T_{\mu\nu}\right)\varepsilon^\nu d^4x=0\,, \tag{2.17}$$

where

$$T_{\mu\nu}=\frac{1}{2}\left(\partial_\mu\Phi\star\partial_\nu\Phi+\partial_\nu\Phi\star\partial_\mu\Phi\right)-g_{\mu\nu}\mathcal{L}\,. \tag{2.18}$$

However we should remind that the divergence of $T_{\mu\nu}$ is not zero, e.g. for the particular case of $V_\star(\Phi)=\lambda/4!\Phi^{\star4}$ we can write:

$$\begin{aligned}\partial_\mu T^{\mu\nu}&=\frac{1}{2}\Big[\Box\Phi\star\partial^\nu\Phi+\partial^\mu\Phi\star\partial_\mu\partial^\nu\Phi+\partial_\mu\partial^\nu\Phi\star\partial^\mu\Phi+\partial^\nu\Phi\star\Box\Phi\Big]-\\&\quad-\frac{1}{2}\Big[\partial^\nu\partial_\mu\Phi\star\partial^\mu\Phi+\partial_\mu\Phi\star\partial^\mu\partial^\nu\Phi\Big]+\frac{m^2}{2}\Big[\partial^\nu\Phi\star\Phi-\Phi\star\partial^\nu\Phi\Big]+\\&\quad+\frac{\lambda}{4!}\Big[\partial^\nu\Phi\star\Phi^{\star3}+\Phi\star\partial^\nu\Phi\star\Phi^{\star2}+\Phi^{\star2}\star\partial^\nu\Phi\star\Phi+\Phi^{\star3}\star\partial^\nu\Phi\Big]\,.\end{aligned} \tag{2.19}$$

Using the equations of motion for the Φ^4 case:

$$\partial_\mu\partial^\mu\Phi+m^2\Phi+\frac{\lambda}{3!}\Phi^{\star3}=0 \tag{2.20}$$

we can rewrite the divergence of the energy-momentum tensor

$$\begin{aligned}\partial_\mu T^{\mu\nu}&=-\frac{\lambda}{2\cdot3!}\Big[\Phi^{\star3}\star\partial^\nu\Phi+\partial^\nu\Phi\star\Phi^{\star3}\Big]+\\&\quad+\frac{\lambda}{4!}\Big[\partial^\nu\Phi\star\Phi^{\star3}+\Phi\star\partial^\nu\Phi\star\Phi^{\star2}+\Phi^{\star2}\star\partial^\nu\Phi\star\Phi+\Phi^{\star3}\star\partial^\nu\Phi\Big]\\&=\frac{\lambda}{4!}\Big[-\partial^\nu\Phi\star\Phi^{\star3}+\Phi\star\partial^\nu\Phi\star\Phi^{\star2}+\Phi^{\star2}\star\partial^\nu\Phi\star\Phi-\Phi^{\star3}\star\partial^\nu\Phi\Big]\\&=\frac{\lambda}{4!}\Big[[\Phi,\partial^\nu\Phi]_{\mathrm{MB}}\star\Phi^{\star2}-\Phi\star2\star[\Phi,\partial^\nu\Phi]_{\mathrm{MB}}\Big]\\&=\frac{\lambda}{4!}\left[[\Phi,\partial^\nu\Phi]_{\mathrm{MB}},\Phi^{\star2}\right]_{\mathrm{MB}}\end{aligned} \tag{2.21}$$

which, of course, along the earlier discussions on the conserved charges is not going to destroy the energy-momentum conservation, for $\theta_{0i} = 0$ cases.

If it happens that the lagrangian density is invariant under some *internal* symmetry, we can compute explicitly the Noether current.

For this we assume that the lagrangian, as in the commutative theory, depends only on Φ and $\partial\,\Phi$ and we will make abstraction of the internal structure of the star product. It is well known that when we vary the lagrangian we find some terms which yield the equation of motion in the lagrangian representation, and also a surface term which will give the Noether current. This surface terms can only come from the kinetic part of the lagrangian. For a term of the form

$$\mathcal{L}_{\rm kin}\big(\Phi(x), \partial\Phi(x)\big) = \mathcal{F}^{\mu}\big(\Phi(x), \partial\Phi(x)\big) \star \partial_{\mu}\Phi\,, \tag{2.22}$$

the corresponding surface term will appear when we vary the $\partial_\mu \Phi$ and the part which enters the conserved current corresponding to this variation is:

$$\mathcal{J}_{\mu} = \mathcal{F}_{\mu} \star \delta\Phi\,. \tag{2.23}$$

Let us consider as a first example a theory with fermions, i.e. QED,

$$\begin{aligned} \mathcal{L}_{\rm kin}\big(\Psi, \bar\Psi\big) \;&=\; \bar\Psi \star (i\gamma_{\mu}\partial_{\mu})\,\Psi \quad \text{with the symmetry:} \\ &\Psi \to e^{i\alpha}\Psi \quad \text{and} \\ &\bar\Psi \to e^{-i\alpha}\bar\Psi\,. \end{aligned} \tag{2.24}$$

Taking Ψ to $\Psi + \delta\Psi$ we can write:

$$\begin{aligned} \delta\mathcal{L}_{\rm kin} &= \bar\Psi \star \big(i\gamma_{\mu}\partial_{\mu}\big)\big(\Psi + \delta\Psi\big) - \bar\Psi \star \big(i\gamma_{\mu}\partial_{\mu}\big)\Psi \\ &= \bar\Psi \star \big(i\gamma_{\mu}\partial_{\mu}\big)\delta\Psi \\ &= \partial_{\mu}\Big(\Psi \star \big(i\gamma_{\mu}\partial_{\mu}\big)\delta\Psi\Big) - \big(\partial_{\mu}\bar\Psi\big) \star \big(i\gamma^{\mu}\delta\Psi\big)\,. \end{aligned} \tag{2.25}$$

For an infinitesimal symmetry transformation $\delta\Psi$ will be:

$$\delta\Psi = \varepsilon\Psi\,, \tag{2.26}$$

so that for a global symmetry the current takes the form:

$$\mathcal{J}_{\mu} = i\bar\Psi \star \big(\gamma_{\mu}\psi\big)\,. \tag{2.27}$$

For the case of local symmetry, we can encounter two types of fermions (say type a and b) and consequently two different symmetry transformations [13, 14]. The arguments we have presented up to now are still valid for a local symmetry so that the conserved currents will be:

$$\mathcal{J}^{a}_{\mu} \;=\; \bar\Psi\gamma_{\mu} \star \Psi \star \varepsilon \tag{2.28a}$$

$$\mathcal{J}^{b}_{\mu} \;=\; \bar\Psi\gamma_{\mu} \star \varepsilon \star\, \Psi\,. \tag{2.28b}$$

JHEP01(2001)025

3. Canonical quantization of noncommutative theories

3.1 Scalar theory

Here we consider scalar theories with arbitrary interaction $V_\star(\Phi)$. The star means that the interaction contains terms with star products, however the precise form of this terms is not important for the general discussion. Let S be the action of our theory:

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 - V_\star(\Phi)\right] . \tag{3.1}$$

Since the free part of the action is identical to the one in the commutative case, it is convenient to choose the Fock space and in particular the vacuum state *to be exactly the same* as in the corresponding commutative theory so, the fields can be expanded in terms of the same (compared to the commutative case) creation and annihilation operators

$$\Phi(x) = \sum_k \left[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}\right] \; e^{i\omega t} . \tag{3.2}$$

For applying the canonical quantization method we should first compute the conjugate momenta $\Pi(x)$ and then impose the quantization conditions

$$[\Phi(\vec{x},t), \Pi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}) . \tag{3.3}$$

However a naive application of this method may lead to severe problems. First as we noticed for the classical theory, in the case $\theta_{0i} \neq 0$ the theory seems to be problematic [4, 25]. That is why we study only the case $\theta_{0i} = 0$. For this case the conjugate momentum is just the usual one which appears in the commutative theory:

$$\Pi = \partial_0\Phi . \tag{3.4}$$

In the commutative case position and momentum space are completely equivalent and we can perform our quantization where we like. However, in the noncommutative theory there is an ambiguity in applying the quantization conditions in the position space. In general we know that in order to deal with a noncommutative space we should work in a usual space and we should replace the products between functions with the star product. But the quantization conditions (3.3) are defined for Φ and Π computed in different points, while the star product makes sense only between functions computed in the same point (see eq. (1.6)). We can escape these problems, if from the very beginning we work in the momentum space and apply directly the quantization conditions in the momentum space:

$$\left[\tilde{\Phi}(k), \tilde{\Pi}(q)\right] = i\delta^{(4)}(k-q) . \tag{3.5}$$

This is possible because in momentum space the difference between the usual commutator and the Moyal bracket is just a phase factor $e^{ik\theta q}$ which has no relevance due to the δ-function which appears in the r.h.s. of eq. (3.5).

From this point the quantization can go on in the same way as in the commutative case. At the level of the free theory everything is the same and only the interaction keeps track of the noncommutative structure of the space through the star product.

3.2 Fermionic theories

For fermions we can apply the same arguments as in the previous section. The free action for fermions reads:

$$S_{free} = \int d^4x \bar{\Psi} \Big(i\gamma^\mu \partial_\mu - m \Big) \Psi \,, \tag{3.6}$$

where $\Psi(x)$ and $\bar{\Psi}(x)$ can be expanded in Fourier modes:

$$\begin{aligned} \Psi(x) &= \sum_k \left[b(k) u(k) e^{-ikx} + d^\dagger(k) v(k) e^{ikx} \right] \; e^{i\omega t} \quad \text{and} \\ \bar{\Psi}(x) &= \sum_k \left[d(k)\, u^\dagger(k)\, e^{-ikx} + b^\dagger(k)\, v^\dagger(k)\, e^{ikx} \right] \; e^{i\omega t} \,. \end{aligned} \tag{3.7}$$

As for the scalar field theory, quantization in position space is ambiguous so we are going to use directly the quantization conditions in momentum space:

$$\Big\{ \psi(k), \psi^\dagger(q) \Big\} = \delta^{(4)}(k-q) \,. \tag{3.8}$$

For the gauge fields, because of the gauge fixing problem, the canonical quantization is more non-trivial, however they are out of the scope of the present work. The only comment we are going to make is that for the gauge theories where the canonical quantization works in the commutative case, the procedure similarly goes through for the noncommutative case.

3.3 Interactions

The next step is to introduce an interaction and derive the Feynman rules. For simplicity we shall restrict ourselves to the scalar theory with Φ^4 interaction, but the arguments can be applied in the same way for other theories.
Let $\phi(k)$ be the Fourier components of Φ:

$$\Phi(x) = \int đ^4 k e^{ikx} \phi(k) \,. \tag{3.9}$$

Then:

$$\begin{aligned}
S_{\rm int} &= \frac{\lambda}{4!}\int d^4x \Phi\star\Phi\star\Phi\star\Phi \qquad (3.10)\\
&= \frac{\lambda}{4!}\int d^4x \,(\Phi\star\Phi)\cdot(\Phi\star\Phi)\\
&= \frac{\lambda}{4!}\int đ^4x \int đ^4k_1\cdots đ^4k_4 e^{-\frac{i}{2}(k_1\theta k_2)}\, e^{-\frac{i}{2}(k_3\theta k_4)}\, e^{i(k_1+k_2+k_3+k_4)\,x}\,\times\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\\
&= \frac{\lambda}{4!}\int đ^4k_1\cdots đ^4k_4 e^{-\frac{i}{2}(k_1\theta k_2)} e^{-\frac{i}{2}(k_3\theta k_4)}\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\,\times\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times(2\pi)^4\delta^{(4)}(k_1+k_2+k_3+k_4)\,.
\end{aligned}$$

Except the exponential, all the factors are symmetric in $k_1\ldots k_4$, hence we get:

$$\begin{aligned}
S_{\rm int} &= \frac{\lambda}{3\cdot 4!}\int đ^4k_1\cdots đ^4k_4\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)(2\pi)^4\delta^{(4)}(k_1+k_2+k_3+k_4)\,\times\\
&\times\left[\cos\frac{k_1\theta k_2}{2}\cos\frac{k_3\theta k_4}{2}+\cos\frac{k_1\theta k_3}{2}\cos\frac{k_2\theta k_4}{2}+\cos\frac{k_1\theta k_4}{2}\cos\frac{k_2\theta k_3}{2}\right]. \qquad (3.11)
\end{aligned}$$

Therefore the only difference which appears in the noncommutative theory, compared to the commutative one, is that for every vertex in noncommutative Φ^4 theory we should multiply by an additional factor:

$$\frac{1}{3}\left[\cos\frac{k_1\theta k_2}{2}\cos\frac{k_3\theta k_4}{2}+\cos\frac{k_1\theta k_3}{2}\cos\frac{k_2\theta k_4}{2}+\cos\frac{k_1\theta k_4}{2}\cos\frac{k_2\theta k_3}{2}\right]. \qquad (3.12)$$

4. Path integral quantization of noncommutative theories

In this section we develop the path integral formulation for noncommutative field theories. Although we specialize to the scalar Φ^4 theory, our method and all the arguments are valid for any noncommutative field theory. Then we give the diagrammatic expression of the effective action up to two loops which we are going to use in the study of the renormalizability of this theory in section 6.

4.1 Measures

For the path integral formalism we should modify the measure of the functional integral according to the convention we are using, i.e. to replace the usual products between functions with the star product

$$(D\Phi\star) = \lim_{N\to\infty} d\Phi(x_1)\star d\Phi(x_2)\star\cdots\star d\Phi(x_n)\,. \qquad (4.1)$$

However in momentum space the star product just introduces a phase factor which in general is going to disappear when we impose the normalization condition for the partition function. So, as far as the measure is concerned, we can forget about the noncommutative structure of the space and work with the usual measure. The fact that the measures in the noncommutative case should be the same as the commutative ones, in the canonical quantization method translates into the point that the *perturbative* Hilbert space for these theories are the same. The same kind of arguments hold for other theories as fermions and gauge fields.[3] In what we are interested, *the perturbation theory*, we can consider that the measure remains unchanged.

4.2 N-point functions and effective action for noncommutative theories

As we emphasized in the previous section, the noncommutative free theory is the same as the commutative one. The only thing that is changed is the interaction. So, we can say that we are dealing with a usual field theory defined on a usual space with the usual Hilbert space, but with strange interactions. For this reason, the noncommutative correlation functions should be defined in the same way as in the commutative theory.

$$G^{(n)}(x_1 \dots x_n) = \langle 0|T(\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_n))|0\rangle \,. \tag{4.2}$$

From which we can conclude that the partition function has the same form as in the usual case,

$$Z[J] = \int (D\Phi)\, e^{iS_{nc}+i\int d^4x J(x)\Phi(x)}\,, \tag{4.3}$$

and from here the generating functional for connected graphs

$$Z[J] = e^{iW[J]}\,, \tag{4.4}$$

and the effective action

$$\begin{aligned} \Gamma[\Phi_c] &= W[J] - \int J(x)\Phi_c(x) d^4x \qquad \text{where} \\ \Phi_c(x) &= \frac{\delta W[J]}{\delta J(x)}\,. \end{aligned} \tag{4.5}$$

At this point we can repeat the calculation from the commutative case in order to find an analytic expression for the effective action:

$$\begin{aligned} 0 &= \int (D\Phi)\left(\frac{\hbar}{i}\right)\cdot\frac{\delta}{\delta\Phi(x)} e^{\frac{i}{\hbar}\left[S_{nc}(\Phi)+\int J(y)\Phi(y)\,dy\right]} \\ &= \int (D\Phi)\left(\frac{\delta S}{\delta\Phi(x)} + J(x)\right) e^{\frac{i}{\hbar}\left[S_{nc}(\Phi)+\int J(y)\Phi(y)\,dy\right]}. \end{aligned} \tag{4.6}$$

[3]However for the gauge fields one should note that the "physical" measure in which ghosts (or gauge degrees of freedom) have been thrown away, is the same for noncommutative and commutative cases. This in turn provides a strong support for the so called Seiberg-Witten map [3] between commutative and noncommutative theories.

To perform the functional integral we should replace, as in the commutative case, $\Phi(x)$ with $\hbar\delta/i\delta J(x)$. However because of the star products which appear in $\delta S/\delta J(x)$, in the noncommutative case this replacement requires more attention. In the following we are going to explain this step in detail for the case of the scalar Φ^4 theory. The only term in $\delta S/\delta J$ which still contains star products is

$$\frac{\delta S_{\rm int}(\Phi)}{\delta\Phi(x)} = -\frac{\lambda}{6}\big(\Phi\star\Phi\star\Phi\big)(x)\,. \tag{4.7}$$

The star product can be expanded in terms with infinite number of partial derivatives, and taking into account that the partial and the functional derivatives commute, we have:

$$\begin{aligned}
&\int (D\Phi)\,\big(\Phi\star\Phi\star\Phi\big)(x)e^{\frac{i}{\hbar}\left[S_{nc}(\Phi)+\int J(y)\Phi(y)\,dy\right]} = \\
&\quad= \Big[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}}\int(D\Phi)\Phi(x+\xi)\Phi(x+\eta+\alpha)\Phi(x+\eta+\beta)\times \\
&\qquad\qquad\times e^{\frac{i}{\hbar}\left[S_{nc}(\Phi)+\int J(y)\Phi(y)\,dy\right]}\Big]_{\{\xi\}=0} \\
&\quad= \Big[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}}\left(\frac{\hbar}{i}\right)^3\frac{\delta}{\delta J(x+\xi)}\frac{\delta}{\delta J(x+\eta+\alpha)}\times \\
&\qquad\qquad\times\frac{\delta}{\delta J(x+\eta+\beta)}e^{\frac{i}{\hbar}W[J]}\Big]_{\{\xi\}=0} \\
&\quad\equiv\left(\frac{\hbar}{i}\right)^3\left(\frac{\delta}{\delta J}\star\frac{\delta}{\delta J}\star\frac{\delta}{\delta J}\right)(x)e^{\frac{i}{\hbar}W[J]}\,,
\end{aligned} \tag{4.8}$$

where the notation $\{\xi\}=0$ means $\xi=\eta=\alpha=\beta=0$. This is a formal way of writing this result in order to make the resemblance with the commutative case more clear, but it is not completely wrong since the functional derivative $\delta F[J]/\delta J(x)$ of a functional F *is a function of* x. With this remark we can write the effective action (for any theory) as in the commutative case:

$$\frac{\delta\Gamma[\Phi_c]}{\delta\Phi_c(x)} = \left(\frac{\delta S}{\delta\Phi(x)}\right)_{\Phi(x)\to\Phi_c(x)+\frac{\hbar}{i}\int G(x,x')\frac{\delta}{\delta\Phi_c(x')}\,d^4x'}\,. \tag{4.9}$$

5. The effective action for the noncommutative Φ^4 theory

As in the usual field theories, we study the effective action and the Green's (two point) function through the power expansion in $\hbar$:

$$\begin{aligned}
\Gamma[\Phi_c] &= \Gamma_0(\Phi_c)+\frac{\hbar}{i}\Gamma_1(\Phi_c)+\left(\frac{\hbar}{i}\right)^2\Gamma_2(\Phi_c)+\cdots \\
G^{ij} &= G_0^{ij}+\frac{\hbar}{i}G_1^{ij}+\left(\frac{\hbar}{i}\right)^2G_2^{ij}+\cdots\,,
\end{aligned} \tag{5.1}$$

where Γ_i and G_i are the i-th order loop corrections.

JHEP01(2001)025

5.1 One loop effective action

Using (4.9) in this subsection we work out the one-loop effective action. As we stressed before, the only thing which is different from the commutative case is the interaction term, so we only consider this term.

$$
\begin{aligned}
\left(\frac{\delta S_{\rm int}}{\delta\Phi(x)}\right)_{\Phi\to\Phi_c+\frac{\hbar}{i}\int G\frac{\delta}{\delta\Phi_c}} &= \frac{\lambda}{6}e^{iW[J]}\left(\Phi_c(x)+\frac{\hbar}{i}\int d^4yG(x,y)\frac{\delta}{\delta\Phi_c(y)}\right)\star \\
&\quad\star\left(\Phi_c(x)+\frac{\hbar}{i}\int d^4yG(x,y)\frac{\delta}{\delta\Phi_c(y)}\right)\star\Phi_c(x) \\
&= -\frac{\lambda}{6}e^{iW[J]}\Big(\Phi_c\star\Phi_c\star\Phi_c\Big)(x) - \\
&\quad -\frac{\lambda}{6}\frac{\hbar}{i}e^{iW[J]}\Phi_c(x)\star\int d^4yG(x,y)\frac{\delta}{\delta\Phi_c(y)}\star\Phi_c(x) - \qquad (5.2a)\\
&\quad -\frac{\lambda}{6}\frac{\hbar}{i}e^{iW[J]}\int d^4yG(x,y)\frac{\delta}{\delta\Phi_c(y)}\star\big(\Phi_c\star\Phi_c\big)(x)+\mathcal{O}(\hbar^2). \qquad (5.2b)
\end{aligned}
$$

The star products in the previous expressions are to be understood as follows:

$$
\begin{aligned}
(5.2a) \propto & \left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}}\Phi_c(x+\xi)\int d^4yG(x+\eta+\alpha,y)\,\frac{\delta}{\delta\Phi_c(y)}\times\right.\\
&\left.\times\,\Phi_c(x+\eta+\beta)\right]_{\{\xi\}=0}\\
=&\left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}}\Phi_c(x+\xi)\int d^4yG(x+\eta+\alpha,y)\,\delta(y-x-\eta-\beta)\right]_{\{\xi\}=0}\\
=&\left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}}\Phi_c(x+\xi)G(x+\eta+\alpha,x+\eta+\beta)\right]_{\{\xi\}=0}\\
=&\left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}}\Phi_c(x+\xi)\int đ^4k\tilde{G}(k)e^{ik(\alpha-\beta)}\right]_{\{\xi\}=0}. \qquad (5.3)
\end{aligned}
$$

In the last expression there is no term which is η dependent so from the expansion of the first exponential we only remain with the first term i.e. 1, while the second exponential will give $e^{\frac{i}{2}k\theta k}$ which is also 1 due to the antisymmetry of θ. So,

$$
\begin{aligned}
(5.2a) \propto \Phi_c(x)\int đ^4k\tilde{G}(k) &= \Phi_c(x)G(0)\\
&= \Phi_c(x)G_0(0)\,. \qquad (5.4)
\end{aligned}
$$

For the second term the calculations go on in the same way and:[4]

$$(5.2b) \propto \left\{ e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}} e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}} \int d^4y G(x+\xi, y) \times \right.$$
$$\left. \times \frac{\delta}{\delta\Phi_c(y)} \Big[\Phi_c(x+\eta+\alpha)\cdot\Phi(x+\eta+\beta) \Big] \right\}_{\{\xi\}=0}$$
$$= G(0)\Phi_c(x) + \int {\mathchar'26\mkern-12mu d}^4k {\mathchar'26\mkern-12mu d}^4q e^{-\frac{i}{2}(k(2\,\theta)q)} e^{iqx} \tilde{G}(k)\phi_c(q)\,. \qquad (5.5)$$

Altogether we can write the one loop contribution to the effective action:

$$\frac{\delta\Gamma_1}{\delta\Phi_c(x)} = -\frac{\lambda}{3} G_0(0)\Phi_c(x) - \frac{\lambda}{6}\int {\mathchar'26\mkern-12mu d}^4k {\mathchar'26\mkern-12mu d}^4q e^{-\frac{i}{2}(k(2\,\theta)q)} e^{iqx} \tilde{G}_0(k)\phi_c(q)\,. \qquad (5.6)$$

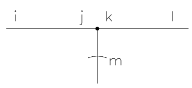
5.2 Diagrammatics

To give the diagrammatic expansion of the effective action, first one should derive the Feynman rules for the noncommutative Φ^4 theory. It will not be surprising to say that the usual rules can be applied. This is because the free theory is the same as in the commutative case. So any line will represent a G_0 and for a vertex with n lines coming out one should write the n-th order functional derivative of the classical action. The argument can go further

$$\frac{\delta S}{\delta\Phi_c(x_3)} G_0(x_1, x_2) = \int d^4y d^4z G_0(x_1, y) \frac{\delta^3}{\delta\Phi(y)\delta\Phi(z)\delta\Phi(x_3)} G_0(z, x_2)\,, \qquad (5.7)$$

or in diagrammatic and condensed notation:

$$\frac{\delta}{\delta\Phi_c^m}\Bigg($$ 

$$\Bigg) =$$ 

. (5.8)

Using these conventions, it is easy to verify that we will recover the Feynman rules we found in the canonical quantization method for noncommutative Φ^4 theory. For this, we have to compute the fourth order functional derivative of the interaction term. This is done in the appendix A and the result written in momentum space (A.8),

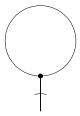
$$\frac{\delta^4 S}{\delta\phi_c(k_1)\delta\phi_c(k_2)\delta\phi_c(k_3)\delta\phi_c(k_4)} \propto \frac{1}{3}\Big[\cos\frac{k_1\theta k_2}{2}\cos\frac{k_3\theta k_4}{2} + 2\text{perm}\Big]\,, \qquad (5.9)$$

is exactly the factor we wrote for the noncommutative vertex (3.12).

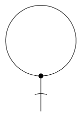
[4]Note that $\phi_c(k)$ are the Fourier modes of $\Phi_c(x)$.

5.3 Diagrammatic expansion of the effective action

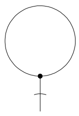
We are now ready to write down the diagrammatic expansion of the effective action for the noncommutative Φ^4 theory. First we show explicitly that up to one loop the diagrammatic expressions for the commutative and noncommutative case coincide. To prove this we show that if we start from

$$\frac{\delta \Gamma_1}{\delta \Phi_c(x)} = \frac{1}{2}$$  $$, \tag{5.10}$$

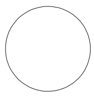
exactly we are going to recover eq. (5.6). Using the diagrammatic rules described in the previous section and the result for the third order functional derivative of S, (A.6) we have:

$$\frac{1}{2}$$ 
$$\begin{aligned} &= \frac{1}{2}\int d^4y d^4z G_0(y,z) \frac{\delta^3 S}{\delta\Phi(y)\delta\Phi(z)\delta\Phi(x)} \\ &= -\frac{1}{2}\frac{\lambda}{6}\Big\{ e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}} e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}} \Big[G(x+\xi, x+\eta+\alpha)\Phi(x+\eta+\beta) + \\ &\quad + G(x+\xi, x+\eta+\beta)\Phi(x+\eta+\alpha) + G(x+\eta+\alpha, x+\xi)\Phi(x+\eta+\beta) + \\ &\quad + G(x+\eta+\beta, x+\xi)\Phi(x+\eta+\alpha) + G(x+\eta+\alpha, x+\eta+\beta)\Phi(x+\xi) + \\ &\quad + G(x+\eta+\beta, x+\eta+\alpha)\Phi(x+\xi)\Big]\Big\}_{\{\xi\}=0} . \end{aligned} \tag{5.11}$$

Written in terms of the Fourier modes (5.11) becomes:

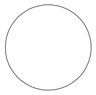
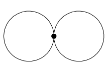
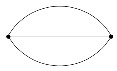
$$\frac{1}{2}$$ 
$$\begin{aligned} &= -\frac{1}{2}\frac{\lambda}{6}\Big\{ e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}} e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}} \int d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q\, \tilde{G}(k)\phi(q) \times \\ &\quad \times \Big[e^{ik(\xi-\eta-\alpha)} e^{iq(x+\eta+\beta)} + e^{ik(\xi-\eta-\beta)} e^{iq(x+\eta+\alpha)} + e^{ik(\eta+\alpha-\xi)} e^{iq(x+\eta+\beta)} + \\ &\quad + e^{ik(\eta+\beta-\xi)} e^{iq(x+\eta+\alpha)} + e^{ik(\alpha-\beta)} e^{iq(x+\xi)} + e^{ik(\beta-\alpha)} e^{iq(x+\xi)} \Big]\Big\}_{\{\xi\}=0} \\ &= -\frac{1}{2}\frac{\lambda}{6}\int d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q\, \tilde{G}(k)\phi(q) e^{iqx}\Big[e^{-\frac{i}{2}(k\theta q)} e^{\frac{i}{2}(k\theta q)} + e^{-\frac{i}{2}(k\theta q)} e^{\frac{i}{2}(q\theta k)} + \\ &\quad + e^{\frac{i}{2}(k\theta q)} e^{-\frac{i}{2}(k\theta q)} + e^{\frac{i}{2}(k\theta q)} e^{-\frac{i}{2}(q\theta k)} + 1 + 1 \Big] \\ &= -\frac{\lambda}{3} G_0(0)\,\Phi_c(x) - \frac{\lambda}{6}\int d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q\, e^{-\frac{i}{2}(k(2\,\theta)q)} e^{iqx} \tilde{G}_0(k)\phi_c(q) . \end{aligned} \tag{5.12}$$

So, up to one loop the diagrammatic expression of the effective action can be written as:

$$\Gamma[\Phi_c] = \bullet + \left(\frac{\hbar}{i}\right)\frac{1}{2}$$  $$. \tag{5.13}$$

5.4 The effective action at higher orders

In the previous subsection we explicitly proved that the diagrammatic expansion up to one loop of the effective action of the noncommutative Φ^4 theory is similar to the commutative case. Now we are going to argue that even at higher orders the effective action should have the same diagrammatic expansion. This is because in the calculations we were doing for the commutative theory, the order in which we were performing the functional integrals was not important, which is still true for the noncommutative case. Hence we can apply the same argument to compute the effective action for higher loops. Therefore, at two loops we have:

$$\Gamma[\Phi_c] = \bullet + \left(\frac{\hbar}{i}\right)\frac{1}{2}$$  $$+ \left(\frac{\hbar}{i}\right)^2\left[\frac{1}{8}\right.$$  $$+ \frac{1}{12}$$  $$\left.\vphantom{\frac{1}{8}}\right] + \mathcal{O}(\hbar^3)\,. \tag{5.14}$$

5.5 Planar and nonplanar diagrams

Here we introduce another way of treating the loop diagrams [8] which turns out to be useful in the discussion of the renormalizability. Up to now we assumed that the noncommutative vertex is unique and is given by eq. (3.12).

The alternative point of view is to say that the order in which the legs appear in the vertices is important and for every distinct ordering we can associate a phase factor such that when we sum up over all the possible orderings we recover the usual factor (3.12). Let us introduce a notation for a generic vertex with N legs numbered in an arbitrary order (say clockwise):

$$V(k_1 \ldots k_N) = \exp\left(\frac{i}{2}\sum_{1\leq i<j\leq N} k_i\theta k_j\right). \tag{5.15}$$

Due to the momentum conservation in vertices, this factor is invariant under cyclic permutations of the legs. As far as only this phase factor is concerned one can deduce some rules so that any graph can be reduced to a generic vertex for which one can write the phase factor using (5.15).

Rule I: An internal line connecting two different vertices can be contracted without changing the overall phase factor associated to the diagram. The important point to keep in mind is *to preserve the order of the lines.*

Rule II: A line starting and ending in the same vertex which is carrying the momentum k can be removed, but we should also introduce a phase factor

$$\delta\varphi = e^{\pm ik\theta p}\,, \tag{5.16}$$

where p is the algebraic sum of the momenta which are inside the loop.

Applying these two rules any graph can be reduced to a generic vertex in which only the external lines of the original graph enter, multiplied by some phase factors (these phases appear when we apply the *Rule II*) which depend on the external, as well as the internal, momenta of the initial graph. It is obvious that for tree level graphs in order to find this phase factor we should apply only the *Rule I*, so at the end the phase factor will depend only on the external momenta. At loop level however we may find some graphs for which the final phase factor also depends on the internal momenta. These are called nonplanar graphs, while those for which the phase factor depends only on the external momenta are called planar graphs.

6. Renormalization of noncommutative Φ^4 theory

In this chapter we study the renormalizability of the noncommutative Φ^4 theory up to two loops. We recall from the previous chapter that in noncommutative theories we encounter two kinds of diagrams which are giving the loop corrections: planar and nonplanar graphs. The planar graphs are the same as the diagrams in the usual commutative theory, the difference consisting in some numerical and external momentum dependent phase factors. For the nonplanar graphs we find some nontrivial phase factors which depend on the loop momenta and which can modify dramatically the result of integration. In this section we are going to show that applying the usual renormalization procedure for the planar graphs, all the other infinities coming from the nonplanar diagrams are going to be canceled out, yielding a finite result, and in this way we prove the renormalizability of the theory up to two loops.

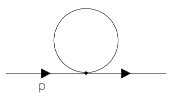
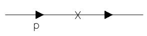
Notation. We are going to extend the attributes planar and nonplanar even on mathematical formulae. A term will be called planar if it does not contain phase factors which depend on the internal momentum, and nonplanar in the other case.

6.1 1-loop renormalization of $\Gamma^{(2)}$

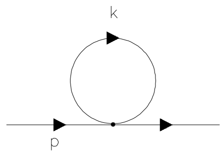
The euclidean action for the noncommutative Φ^4 theory including the counter-terms can be written as:

$$\begin{aligned} S = & \int d\!\!{}^-{}^4x \left[\frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi + \frac{m^2}{2}\,\Phi^2 + \frac{\lambda}{4!}\Phi\star\Phi\star\Phi\star\Phi\right] + \\ & + \int d^4x \left[\frac{1}{2}(Z_3-1)\,\partial_\mu\Phi\,\partial^\mu\Phi + \frac{\delta m^2}{2}\,\Phi^2 + \frac{\delta\lambda}{4!}\Phi\star\Phi\star\Phi\star\Phi\right] . \end{aligned} \tag{6.1}$$

This leads to the following diagrammatic expansion of $\Gamma^{(2)}$ at one loop:

$$\Gamma^{(2)} = \quad$$  $$+\ \frac{1}{2}$$  $$+$$  $$. \tag{6.2}$$

The one loop mass correction in the noncommutative theory has the form:



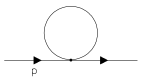
$$= -\frac{\lambda}{3}\int\left[2\left(\cos\frac{p\theta k}{2}\right)^2 + 1\right]\frac{d\!\!{}^-{}^4k}{k^2+m^2}$$

$$
\begin{aligned}
&= -\frac{\lambda}{3}\int \frac{\cos p\theta k + 2}{k^2+m^2}\,\bar{d}^4k \\
&= -\frac{2}{3}\lambda \int \frac{\bar{d}^4k}{k^2+m^2} - \frac{\lambda}{3}\int \frac{\cos p\theta k}{k^2+m^2}\,\bar{d}^4k\,. \qquad (6.3)
\end{aligned}
$$

Using Schwinger parameterization

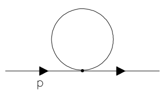
$$
\frac{1}{k^2+m^2} = \int_0^\infty d\alpha\, e^{-\alpha(k^2+m^2)}\,, \qquad (6.4)
$$

the integral over k becomes just an ordinary gaussian integral which can be performed explicitly.



$$
\begin{aligned}
&= -\frac{\lambda}{3}\int_0^\infty d\alpha \int \bar{d}^4k\,\left[2\,e^{-\alpha(k^2+m^2)} + e^{ik\theta p-\alpha(k^2+m^2)}\right] \\
&= -\frac{\lambda}{3(2\pi)^4}\int_0^\infty d\alpha \left(\sqrt{\frac{\pi}{\alpha}}\right)^4 \left[2\,e^{-\alpha m^2} + e^{-\alpha m^2 - \frac{p\circ p}{4\alpha}}\right], \qquad (6.5)
\end{aligned}
$$

in which we have introduced the notation $p\circ k \equiv p\theta\theta k = p_\mu \theta^{\mu\rho}\theta_\rho^{\ \nu} k_\nu$. The integral over α can be regularized by introducing a factor $e^{-1/4\alpha\Lambda^2}$, where Λ plays the role of a cut-off.



$$
\begin{aligned}
\Big[\quad\Big]_{\mathrm{NP}} &= -\frac{\lambda}{3\cdot 2^4\,\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\alpha m^2 - \frac{1}{4\alpha\Lambda_{\mathrm{eff}}^2}} \\
&= -\frac{\lambda}{12\pi^2}\, m^2 \sqrt{\frac{\Lambda_{\mathrm{eff}}^2}{m^2}}\, K_1\left(\frac{m}{\Lambda_{\mathrm{eff}}}\right)\,, \qquad (6.6)
\end{aligned}
$$

where $\Lambda_{\mathrm{eff}}^{-2} = p\circ p + 1/\Lambda^2$, and K_1 is the modified Bessel function [26]. It can be seen that in the limit $\Lambda \to \infty$, Λ_{eff} becomes $p\circ p$ so the integral remains finite regulated by the cosine factor which appears in (6.3).

For the planar part of the diagram we can repeat the calculations with the change $\Lambda_{\mathrm{eff}} \to \Lambda$, and then we absorb the regulated integral in δm^2 to make $\Gamma^{(2)}$ finite

$$
\delta m_1^2 = \frac{\lambda}{3}\int_\Lambda \frac{\bar{d}^4k}{k^2+m^2}\,. \qquad (6.7)
$$

Here we note that the numeric factor, $1/3$, is different from that of the commutative case (which is one). Therefore, considering the loop effects, the $\theta \to 0$ limit is not a smooth limit, and we are not going to recover the usual commutative theory.

We can write now the quadratic part of $\Gamma^{(2)}$ at one loop:

$$
\Gamma_1^{(2)} = \int d^4p \frac{1}{2}\left[p^2 + M^2 + \frac{\lambda}{96\,(2\pi)^2} K_1\left(\frac{M^2}{\Lambda_{\mathrm{eff}}}\right) + \mathcal{O}(\lambda^2)\right]\phi(p)\phi(-p)\,. \qquad (6.8)
$$

Here M^2 is the renormalized mass $M^2 = m^2 + \delta m_1^2$.

For small arguments K_1 can be expanded in Laurent series

$$K_1(z) \xrightarrow{z \to 0} \frac{1}{z} + \frac{z}{2} \ln \frac{z}{2}\,, \tag{6.9}$$

so that for $\Lambda_{\text{eff}} \gg 1$ the quadratic part of the renormalized effective action is:

$$\Gamma^{(2)}_{\text{ren}}(\Lambda) = \int d^4p \frac{1}{2} \phi(p)\phi(-p) \Bigg[p^2 + M^2 + \frac{\lambda}{96\,(2\pi)^2\,(p \circ p + 1/\Lambda^2)} - \frac{\lambda M^2}{96\,\pi^2} \times \\ \times \ln \left(\frac{1}{M^2\,(p \circ p + 1/\Lambda^2)} \right) + \emptyset(\lambda^2) \Bigg]. \tag{6.10}$$

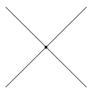
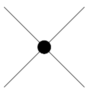
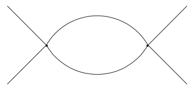
As we see after sending Λ to infinity the $\Gamma^{(2)}$ presents an infrared divergence. Moreover, if we first consider the zero momentum limit the cut-off effect of the non-commutativity cannot be seen any more and the two-point effective action diverges. This is the interesting UV-IR mixing which appears in the noncommutative theories. This divergence can be explained assuming that (6.10) is the wilsonian effective action obtained by integrating out some other field χ (see [7]). Then the action which correctly reproduces the factor $1/p \circ p$ in eq. (6.10) is:

$$\Gamma'(\lambda) = \Gamma(\Lambda) + \int d^4x \left(\frac{1}{2} \partial\chi \circ \partial\chi + \frac{1}{2} \Lambda^2 (\partial \circ \partial\chi)^2 + i\sqrt{\frac{\lambda}{96\pi^2}} \lambda\chi\phi \right) . \tag{6.11}$$

However the logarithmic term in (6.10) is yet to be explained in some other way [17].

6.2 1-loop renormalization of $\Gamma^{(4)}$

The one loop diagrammatic expansion of $\Gamma^{(4)}$ is:

$\Gamma^{(4)} =$  $+$  $+ \frac{1}{2} \Big($  $+ 2\,\text{permutations} \Big)$. (6.12)

The nonplanar part of the loop graph give rise to an integral of the form:[5]

$$I_{np} \equiv \int d\!\!{}^-{}^4k \frac{e^{ip\theta k}}{(k^2 + m^2)\big((p_1 + p_2 - k)^2 + m^2\big)}\,, \tag{6.13}$$

where p is the total momentum which is crossing the loop. As we showed for $\Gamma^{(2)}$, the exponential factor acts as a regulator and the integral remains finite. Moreover, the integral can be evaluated analytically using first Feynman parameterization in order to write the denominator as a square and then using Schwinger parameterization in order to compute the integral over the internal momentum and the final result is:

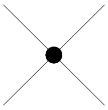
$$I_{np} = \frac{1}{8\pi^2} \int_0^1 dx e^{i\left(p\theta(p_1+p_2)\right)\,x} K_0 \left(\sqrt{\Big[(p_1 + p_2)^2\, x(1 - x) + m^2\Big] p \circ p} \right) . \tag{6.14}$$

[5]The exact phase factors which enter the integral can be found in appendix B

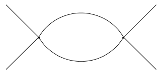
There is still the peculiar IR divergence, but in the following we are going to study only the UV behavior of the theory. So the infinity comes only from the planar part and it is of the form:

$$\frac{\lambda^2}{9}\int \frac{d\!\!{}^-{}^4k}{(k^2+m^2)\big((p_1+p_2-k)^2+m^2\big)} \cos p_1\theta p_2 \cos p_3\theta p_4 + 2\text{permutations}\,. \tag{6.15}$$

Renormalization of $\Gamma^{(4)}$ requires to absorb the infinity in the corresponding counter-term in eq. (6.12), i.e.



$$\Bigg|_{p=0} + \frac{3}{2}\text{Planar}\Bigg[$$



$$\Bigg]\Bigg|_{p=0} = 0$$

$$\Rightarrow \delta\lambda_1 + \frac{3}{2}\cdot\frac{2\lambda^2}{9}\int_\Lambda \frac{d\!\!{}^-{}^4k}{(k^2+m^2)^2} = 0$$
$$\Rightarrow \delta\lambda_1 = -\frac{\lambda^2}{3}\int_\Lambda \frac{d\!\!{}^-{}^4k}{(k^2+m^2)^2}\,. \tag{6.16}$$

We again note that the difference in the numeric factor of $-1/3$ in the above equation compared to the commutative one which is $-3/2$.

At this point using the low external momenta limit we can explicitly write the renormalized effective action at one loop. For small arguments, the modified Bessel function behaves like:

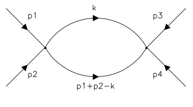
$$K_0(x) \xrightarrow{x\to 0} -\ln\frac{x}{2} + \text{finite}\,, \tag{6.17}$$

so we can write:

$$I_{np} \sim \frac{1}{16\pi^2}\ln\frac{4}{m^2 p\circ p}\,. \tag{6.18}$$

Now,

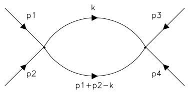
$$\Bigg[\frac{1}{2}$$



$$\Bigg]_{\substack{\text{NP}\\ p_i\to 0}} = \frac{\lambda^2}{9}\int d\!\!{}^-{}^4k \frac{\mathcal{P}(p_1\ldots p4,k,\theta)}{(k^2+m^2)\big((p_1+p_2-k)^2+m^2\big)}\,, \tag{6.19}$$

where $\mathcal{P}$ is given in (B.2).

In the limit which we are considering one can neglect the cosine factors which depend only on the external momenta. We should also notice that I_{NP} does not depend on the sign in the exponential and so taking into account the "2 permutations" from eq. (6.12) we can write:

$$\Bigg[\frac{1}{2}$$


$$+\,2\mathrm{perm}\Bigg]_{\substack{\mathrm{NP}\\ p_i\to 0}} = \frac{\lambda^2}{9}\int d\!\!^{-}{}^4k\,\frac{1}{(k^2+m^2)\big((p_1+p_2-k)^2+m^2\big)}\times$$

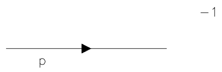
$$\times\left[\sum_{i=2}^{4}e^{ik\theta(p_1+p_i)}+\frac{3}{2}\sum_{i=1}^{4}e^{ik\theta p_i}+\frac{2}{4}\sum_{i=2}^{4}e^{ik\theta(p_1+p_i)}\right]. \tag{6.20}$$

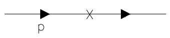
Now using (6.18) we can write the renormalized four point function at one loop:

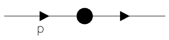
$$\Gamma^{(4)}_{\mathrm{ren}}(p_1\ldots p_4)=\lambda-\frac{\lambda^2}{96\,\pi^2}\Bigg\{\ln\frac{1}{m^2p_1\circ p_1}+\ln\frac{1}{m^2p_2\circ p_2}+\ln\frac{1}{m^2p_3\circ p_3}+ \tag{6.21}$$
$$+\ln\frac{1}{m^2p_4\circ p_4}+\ln\frac{1}{m^2(p_1+p_2)\circ(p_1+p_2)}+$$
$$+\ln\frac{1}{m^2(p_1+p_3)\circ(p_1+p_3)}+\ln\frac{1}{m^2(p_1+p_4)\circ(p_1+p_4)}\Bigg\}.$$

6.3 $\Gamma^{(2)}$ at two loops

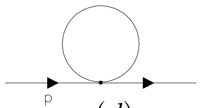
After the one loop calculation, we can proceed to that of two loops. First we start with two point function:

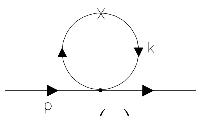
$\Gamma^{(2)} =$


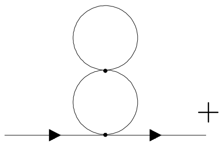
(a) $+$


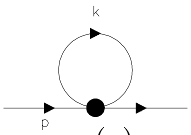
(b) $+$


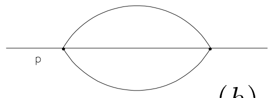
(c) $+$

$+\,\frac{1}{2}$


(d) $+\,\frac{1}{2}$


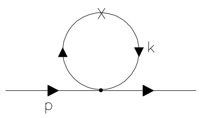
(e) $+\,\frac{1}{4}$

(f) $+$

$+\,\frac{1}{2}$


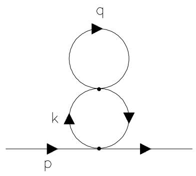
(g) $+\,\frac{1}{6}$


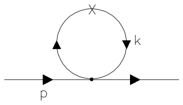
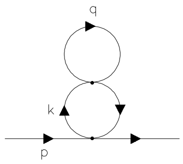
(h) . (6.22)

In the usual commutative Φ^4 theory at two loops the UV divergent parts of terms (e) and (f) cancel out. However, since in the 1-loop mass correction only the planar part have been taken into account (see eq. (6.7)), in the noncommutative theory we are left with another term.

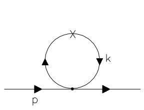
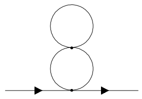
$\frac{1}{2}$


$$=\frac{\lambda^2}{18}\int d\!\!^{-}{}^4k\,d\!\!^{-}{}^4q\,\frac{\cos k\theta p+2}{(q^2+m^2)(k^2+m^2)^2}$$

$$\frac{1}{4}$$  $$= -\frac{\lambda^2}{36}\int d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q \frac{(\cos k\theta p + 2)(\cos k\theta q + 2)}{(q^2+m^2)(k^2+m^2)^2}$$

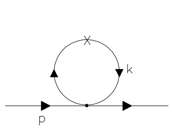
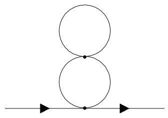
$$\frac{1}{2}$$  $$+\frac{1}{4}$$  $$= -\frac{\lambda^2}{36}\int d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q \frac{\cos k\theta q(\cos k\theta p + 2)}{(q^2+m^2)(k^2+m^2)^2}\,. \tag{6.23}$$

Using Schwinger parameterization, the integral over q can be done explicitly and we remain with:

$$\frac{1}{2}$$  $$+\frac{1}{4}$$  $$= -\frac{\lambda^2}{36}\int d\!\!{}^-{}^4k \int_0^\infty \frac{d\alpha}{(2\pi)^4}\left(\sqrt{\frac{\pi}{\alpha}}\right)^4 \times$$

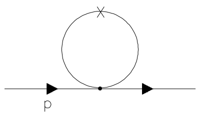
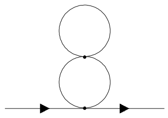
$$\times e^{-\alpha m^2 - \frac{1}{4\alpha\Lambda_{\text{eff}}}} \frac{\cos p\theta k + 2}{(k^2+m^2)^2}\,, \tag{6.24}$$

where Λ_{eff} is given by $\Lambda_{\text{eff}}^{-2} = k \circ k + 1/\Lambda^2$. The integral over α can be exactly computed, yielding a modified Bessel function K_1 and the final result is:

$$\frac{1}{2}$$  $$+\frac{1}{4}$$  $$=$$

$$= -\frac{\lambda^2 m^2}{36}\frac{1}{2^4\pi^2}\int d\!\!{}^-{}^4k \frac{4}{\sqrt{m^2(k\circ k + 1/\Lambda^2)}} K_1\left(\sqrt{\frac{m^2}{\Lambda_{\text{eff}}^2}}\right)\frac{cos p\theta k + 2}{(k^2+m^2)^2}\,. \tag{6.25}$$

Simple power counting tells us that the integration over the large values of k is finite provided K_1 does not diverge at infinity. In fact K_1 decays exponentially at infinity, so the convergence is guaranteed. On the other hand the integration over small values of k can be controlled if we do not let Λ to go to infinity. It is worth noting that in the massless limit ($m \to 0$) all these arguments again holds. In fact in the $m \to 0$ limit all the mass dependence is removed from (6.25). Under these assumptions we can write:

$$\frac{1}{2}$$  $$+\frac{1}{4}$$  $$= \text{finite}\,. \tag{6.26}$$

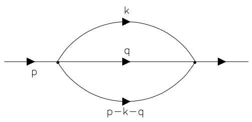
In the usual Φ^4 theory, for the remaining loop diagrams in eq. (6.22), the momentum independent infinities are absorbed in δm^2, while the momentum dependent ones are absorbed in the wave function renormalization.

In noncommutative Φ^4 theory, the momentum dependent factors which appear in the vertices for (h) in eq. (6.22) are:

$$\frac{1}{9}\left[\cos\frac{p\theta k}{2}\cos\frac{q\theta(p-k)}{2} + \cos\frac{p\theta q}{2}\cos\frac{k\theta(p-q)}{2} + \cos\frac{k\theta q}{2}\cos\frac{p\theta(k+q)}{2}\right]^2. \tag{6.27}$$

Expanding both the square and the cosine factors we get:

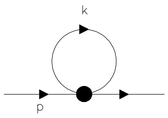
$$\frac{1}{6}$$



$$
\begin{aligned}
&= -\frac{\lambda^2}{36}\int \frac{\text{đ}^4 k\,\text{đ}^4 q}{(k^2+m^2)(q^2+m^2)((p-k-q)^2+m^2)} \times \\
&\quad \times\Big\{ 1 + + \frac{2}{3}\Big[\cos p\theta k + \cos p\theta q + \cos p\theta(k+q)\Big] + \\
&\quad + \frac{2}{3}\Big[\cos k\theta(p-q) + \cos q\theta(p-k) + \cos k\theta q\Big] + \\
&\quad + \frac{1}{3}\Big[\cos\big(p\theta k + q\theta(p-k)\big) + \cos\big(p\theta k - q\theta(p-k)\big) + \\
&\qquad + \cos\big(p\theta k - q\theta(p+k)\big)\Big]\Big\} \\
&= -\frac{\lambda^2}{36}\int \frac{\text{đ}^4 k\,\text{đ}^4 q}{(k^2+m^2)(q^2+m^2)((p-k-q)^2+m^2)} \times \\
&\qquad \times\Big[1 + 2\cos p\theta k + 2\cos k\theta q + \cos\big(p\theta k + q\theta(p-k)\big)\Big].
\end{aligned}
\tag{6.28}
$$

The contribution of the counter-term (g) in eq. (6.22) is:

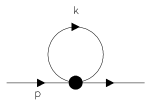
$$\frac{1}{2}$$



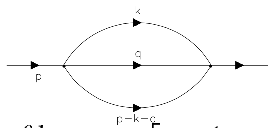
$$
\begin{aligned}
&= -\frac{\lambda^2}{2}\left(-\frac{1}{3}\right)\int \frac{\text{đ}^4 k}{k^2+m^2}\frac{2+\cos p\theta k}{3}\int \frac{\text{đ}^4 q}{(q^2+m^2)^2} \\
&= \frac{\lambda^2}{9}\int \frac{\text{đ}^4 k\,\text{đ}^4 q}{(k^2+m^2)(q^2+m^2)^2} + \frac{\lambda^2}{18}\int \text{đ}^4 k\,\text{đ}^4 q \frac{\cos p\theta k}{(k^2+m^2)(q^2+m^2)^2}.
\end{aligned}
\tag{6.29}
$$

For the planar diagrams we should follow the same renormalization procedure as in the commutative case. More precisely there are new divergences arising from the planar parts of the sum of the above two graphs which should be absorbed in the relevant two loop counter-term. We again note that this counter-term is not θ dependent. We are then left with the nonplanar part which can be written as:

$$\Bigg[\frac{1}{2}$$



$$+\frac{1}{6}$$



$$\Bigg]_{\mathrm{NP}} =$$

$$
\begin{aligned}
&= \frac{\lambda^2}{18}\int \text{đ}^4 k\,\text{đ}^4 q \frac{\cos p\theta k}{(k^2+m^2)(q^2+m^2)}\left[\frac{1}{q^2+m^2} - \frac{1}{(p-k-q)^2+m^2}\right] - \\
&-\frac{\lambda^2}{36}\int \frac{\text{đ}^4 k\,\text{đ}^4 q}{(k^2+m^2)(q^2+m^2)((p-k-q)^2+m^2)}\Big[2\cos k\theta q + \cos\left(p\theta k + q\theta(p-k)\right)\Big].
\end{aligned}
\tag{6.30}
$$

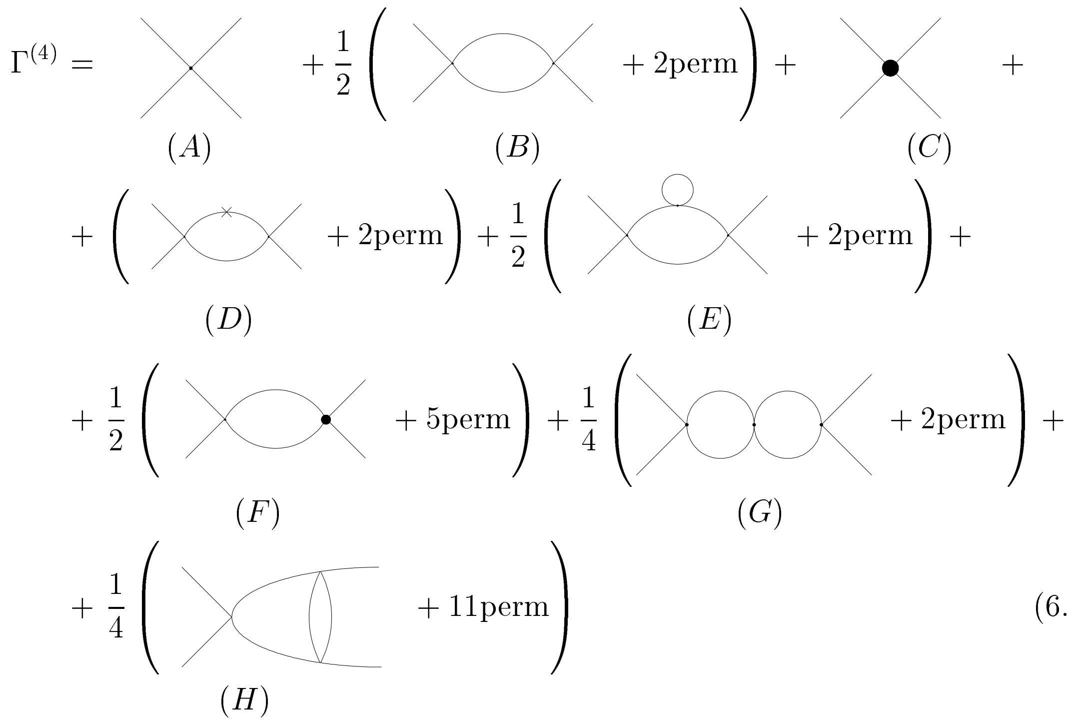
For the first term, the integral over q yields a finite result (this can be seen by simple power counting), while the term $\cos p\theta k$ acts as a regulator for the integral over k. In the second term when we take $\cos k\theta q$ from the square bracket and integrate over q, we get a modified Bessel function $(K(\sqrt{k \circ k}))$ which exponentially decay at infinity and takes care of the integration over large values of q. When we take $\cos(p\theta k + q\theta(p-k))$ by a change of variables ($k' = p - k$ and $q' = p - q$) we can put the integral in the form:

$$\int \frac{d^4k d^4q}{((p-k)^2+m^2)((p-q)^2+m^2)((p-k-q)^2+m^2)} \cos k\theta q$$

for which, in the UV-limit we can apply the same argument as before.

At this point we have proved that renormalizing the planar part of the diagrams appearing in the noncommutative version of the Φ^4 theory, as in the usual case we can make $\Gamma^{(2)}$ finite without renormalizing any other parameter, in particular, θ.

6.4 $\Gamma^{(4)}$ at two loops

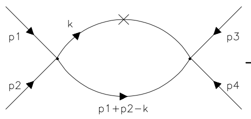


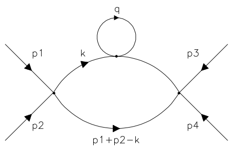
(6.31)

This formula requires some comments. The last term (or the fish diagram) appears twelve times according to the number of permutations of the external momenta which give different contributions. In the commutative case however we can see only six independent permutations. This difference comes from the fact that in the noncommutative theory there are momentum dependent phase factors which appear in vertices, and these factors allow us to distinguish between the last two legs of the fish diagram. Since all we are doing is to take the fourth order functional derivative of the effective action, and the order in which we perform the derivatives has no importance, in the end when we sum up the diagrams coming from all the permutations we should find that the result is invariant under arbitrary relabeling of external momenta. This is the reason why we need 11 permutations in the last term of eq. (6.31).

However, since not all the terms in the fish diagram break explicitly the symmetry between the last legs, we shall consider for simplicity only 5 permutations, but in the end we should remember to symmetrize over the last two momenta.

In the commutative case divergences coming from terms (D) and (E) of eq. (6.31) cancel each other. In the noncommutative case, because of changes in the numeric factors of counter-terms, should be checked again. Using the notation we introduced in eq. (B.1) for the cosine factors appearing in the 1-loop vertex correction, and also the definition of δm_1^2 from eq. (6.7), we can write:

$\Bigg($ 

$+\, 2\text{perm}\Bigg) + \dfrac{1}{2}\Bigg($ 

$+\, 2\text{perm}\Bigg) =$

$$
\begin{aligned}
&= 2\frac{\lambda^2}{9}\int d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q \frac{\mathcal{P}(k,p,\theta)}{(q^2+m^2)(k^2+m^2)^2((p-k)^2+m^2)}\cdot\frac{\lambda}{3} - \\
&\quad -\frac{\lambda^2}{9}\int d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q \frac{\mathcal{P}(k,p,\theta)}{(q^2+m^2)(k^2+m^2)^2((p-k)^2+m^2)}\cdot\frac{\lambda\ (cosk\theta q+2)}{3} \\
&= -\frac{\lambda^3}{27}\int d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q \frac{\cos k\theta q}{q^2+m^2}\cdot\frac{\mathcal{P}(k,p,\theta)}{(k^2+m^2)^2((p-k)^2+m^2)}\,.
\end{aligned}
\tag{6.32}
$$

The q integral is regulated by $\cos k\theta q$, while the integral over k is convergent right from the beginning. This means that even though the sum of these diagrams is nonzero at least it is finite, and this is what we are interested in.

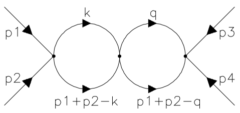
The planar part of the diagrams in (6.31) does not come with anything new, except for some numeric and phase factors which depend only on the external momenta. Nevertheless in order to apply the usual renormalization procedure we should check explicitly that the external momentum dependent factor is the same for all the diagrams which appear in the expansion of $\Gamma^{(4)}$ and this should be exactly the additional phase factor for a noncommutative vertex, i.e.

$$
\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}+\cos\frac{p_1\theta p_3}{2}\cos\frac{p_2\theta p_4}{2}+\cos\frac{p_1\theta p_4}{2}\cos\frac{p_2\theta p_3}{2}\,. \tag{6.33}
$$

The whole calculation with the explicit cosine expansion is given in the appendix B.

Due to the internal momentum phase factors, the nonplanar diagrams are less divergent than the corresponding planar ones. However divergences may still appear whenever the cosine factors do not contain any of the loop momenta. In the following we are going to show that these infinities coming from the nonplanar graphs are going to cancel so that in order to obtain an overall finite result it is enough to renormalize the planar diagrams.

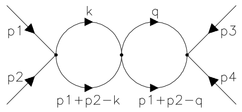
The nonplanar divergent parts of the diagrams (G), (H) in (6.31) are obtained by simply taking the terms which we denoted in the appendix B as "nonplanar k/q independent terms". Keeping track of all the numerical factors we find for the diagram (G):

$$\frac{1}{4}$$ 

$$\Big|_{\text{div. NP}} =$$

$$
\begin{aligned}
&= \frac{\lambda^3}{27} \int \frac{d\!\!{}^-\!{}^4k\, d\!\!{}^-\!{}^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)((p_1+p_2-q)^2+m^2)} \times \\
&\quad \times \left\{ \frac{1}{2} \cos\frac{p_1\theta p_2}{2} \cos\frac{p_3\theta p_4}{2} \left[\cos k\theta(p_1+p_2) + \cos q\theta(p_1+p_2) \right] + \right. \\
&\quad + \frac{1}{4} \cos\frac{p_1\theta p_2}{2} \left[\cos\left(\frac{p_3\theta p_4}{2} - q\theta p_3 \right) + \cos\left(\frac{p_3\theta p_4}{2} + q\theta p_4 \right) \right] + \\
&\quad \left. + \frac{1}{4} \cos\frac{p_3\theta p_4}{2} \left[\cos\left(\frac{p_1\theta p_2}{2} + k\theta p_1 \right) + \cos\left(\frac{p_1\theta p_2}{2} - k\theta p_2 \right) \right] \right\}. \qquad (6.34)
\end{aligned}
$$

Since the propagators corresponding to this diagram are $q \longleftrightarrow k$ symmetric, by a change of variables q can be replaced by k inside the cosine factors.

$$\frac{1}{4}$$ 

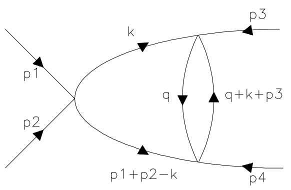
$$\Big|_{\text{div. NP}} =$$

$$
\begin{aligned}
&= \frac{\lambda^3}{27} \int \frac{d\!\!{}^-\!{}^4k\, d\!\!{}^-\!{}^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)((p_1+p_2-q)^2+m^2)} \times \\
&\quad \times \left\{ \cos\frac{p_1\theta p_2}{2} \cos\frac{p_3\theta p_4}{2} \cos k\theta(p_1+p_2) + \frac{1}{4} \cos\frac{p_1\theta p_2}{2} \times \right. \\
&\quad \times \left[\cos\left(\frac{p_3\theta p_4}{2} - k\theta p_3 \right) + \cos\left(\frac{p_3\theta p_4}{2} + k\theta p_4 \right) \right] + \\
&\quad \left. + \frac{1}{4} \cos\frac{p_3\theta p_4}{2} \left[\cos\left(\frac{p_1\theta p_2}{2} + k\theta p_1 \right) + \cos\left(\frac{p_1\theta p_2}{2} - k\theta p_2 \right) \right] \right\}. \qquad (6.35)
\end{aligned}
$$

Now we should do the same with the fish diagram. However here we should notice that the propagators in diagram (H)[6] come with a k^6, so the integration over k is already UV finite despite of the fact that there is no regulator on the k integral coming from the noncommutativity. This means that the nonplanar k-independent terms will give a finite result because they are regulators for the q-integral. So the only nonplanar terms which are contributing to the divergences of the fish diagram

[6] See the picture below for the right assignment of momenta.

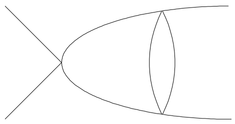
come only from the so called q-independent terms and this can be written:

$\frac{1}{2}$ 

$\Big|_{\text{div NP}} =$

$$= \frac{\lambda^3}{27}\int \frac{d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)((k+q+p_3)^2+m^2)} \times$$
$$\times \frac{1}{4}\Bigg\{\cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2} + k\theta(p_1+p_2)\right) + \cos\left(\frac{p_1\theta p_2 + p_3\theta p_4}{2} - k\theta(p_1+p_2)\right) +$$
$$+\cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2} + k\theta p_1\right) + \cos\left(\frac{p_1\theta p_2 + p_3\theta p_4}{2} - k\theta p_2\right) +$$
$$+\cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2} + k\theta(p_1+p_3)\right) + \cos\left(\frac{p_1\theta p_2 + p_3\theta p_4}{2} + k\theta(p_1+p_4)\right) +$$
$$+ 2\cos\frac{p_1\theta p_2}{2}\left[\cos\left(\frac{p_3\theta p_4}{2} - k\theta p_3\right) + \cos\left(\frac{p_3\theta p_4}{2} + k\theta p_4\right)\right]\Bigg\}. \qquad (6.36)$$

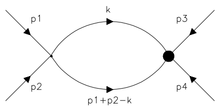
The factor in front of the fish diagram is $1/2$ because as explained at the beginning of this section we are considering only five permutations instead of eleven and we are going to symmetrize the result with respect to p_3 and p_4 in the end. So the contribution of the fish diagram should be written:

$\frac{1}{2}$  $\Big|_{\text{div NP}} =$

$$= \frac{\lambda^3}{27}\int \frac{d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)((k+q+p_3)^2+m^2)} \times$$
$$\times \frac{1}{4}\Bigg\{\frac{1}{2}\Bigg[\cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2} + k\theta(p_1+p_2)\right) +$$
$$+\cos\left(\frac{p_1\theta p_2 + p_3\theta p_4}{2} - k\theta(p_1+p_2)\right) + p_3 \leftrightarrow p_4\Bigg] +$$
$$+\frac{1}{2}\left[\cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2} + k\theta p_1\right) + \cos\left(\frac{p_1\theta p_2 + p_3\theta p_4}{2} - k\theta p_2\right) + p_3 \leftrightarrow p_4\right] +$$
$$+\cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2} + k\theta(p_1+p_3)\right) + \cos\left(\frac{p_1\theta p_2 + p_3\theta p_4}{2} + k\theta(p_1+p_4)\right) +$$
$$+ 2\cos\frac{p_1\theta p_2}{2}\left[\cos\left(\frac{p_3\theta p_4}{2} - k\theta p_3\right) + \cos\left(\frac{p_3\theta p_4}{2} + k\theta p_4\right)\right]\Bigg\}$$
$$= \frac{\lambda^3}{27}\int \frac{d\!\!{}^-{}^4k\, d\!\!{}^-{}^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)((k+q+p_3)^2+m^2)} \times$$

$$
\times \frac{1}{4}\Bigg\{ 2\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\cos k\theta(p_1+p_2) + \cos\frac{p_3\theta p_4}{2}\times
$$
$$
\times\left[\cos\left(\frac{p_1\theta p_2}{2}+k\theta p_1\right) + +\cos\left(\frac{p_1\theta p_2}{2}-k\theta p_2\right)\right] +
$$
$$
+2\cos\frac{p_1\theta p_2}{2}\left[\cos\left(\frac{p_3\theta p_4}{2}-k\theta p_3\right)+\cos\left(\frac{p_3\theta p_4}{2}+k\theta p_4\right)\right]
$$
$$
+\cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2}+k\theta(p_1+p_3)\right)+\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+k\theta(p_1+p_4)\right)\Bigg\},
\tag{6.37}
$$

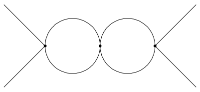
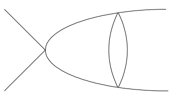
and now we have truly only 5 more permutations. Up to now we have just discussed the two loop diagrams which have a divergent nonplanar part. Some more divergences which can be classified as nonplanar ones come from the counterterm denoted by (F) in (6.31). These terms can be written as follows:

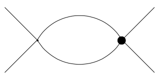
$\frac{1}{2}$ 

$\Bigg|_{\text{nonplanar}} =$

$$
= -\frac{\lambda^3}{27}\int \frac{đ^4k\, đ^4q}{((p_1+p_2-k)^2+m^2)(k^2+m^2)(q^2+m^2)^2}\times
$$
$$
\times\Bigg\{\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\cos k\theta(p_1+p_2)+\frac{1}{2}\cos\frac{p_1\theta p_2}{2}\times
$$
$$
\times\left[\cos\left(\frac{p_3\theta p_4}{2}-k\theta p_3\right)+\cos\left(\frac{p_3\theta p_4}{2}+k\theta p_4\right)\right]+
$$
$$
+\frac{1}{4}\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+k\theta(p_1+p_4)\right)+
$$
$$
+\frac{1}{4}\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}-k\theta(p_1+p_3)\right)+\frac{1}{2}\cos\frac{p_3\theta p_4}{2}\times
$$
$$
\times\left[\cos\left(\frac{p_1\theta p_2}{2}+k\theta p_1\right)+\cos\left(\frac{p_1\theta p_2}{2}-k\theta p_2\right)\right]\Bigg\}.
\tag{6.38}
$$

Now we have to combine the results found for the divergent nonplanar pieces in (6.35), (6.37) and (6.38) and show that at the end of the day we remain only with a finite piece. The quantity we are dealing with is:

$\Bigg[\frac{1}{4}\Bigg($  $+ 2\text{perm}\Bigg) + \frac{1}{2}\Bigg($  $+ 5\text{perm}\Bigg)\Bigg]_{\text{div. NP}} +$

$+\Bigg[\frac{1}{2}\Bigg($  $+ 5\text{perm}\Bigg)\Bigg]_{\text{NP}}, \qquad (6.39)$

and in order to show that this is finite we will check separately each independent cosine combination. Let us start with $\cos(p_1\theta p_2/2)\cos(p_3\theta p_4/2)\cos k\theta(p_1+p_2)$. This term can be found in all the three pieces of (6.39). The last two graphs come together with five permutations, but due to the conservation of momenta i.e. $p_1+p_2=-(p_3+p_4)$ only two out of the five permutations are independent:

$$\begin{aligned}
&\frac{1}{2}\left[\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\cos k\theta(p_1+p_2)+5\text{ permutations}\right]= \\
&\quad=\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\cos k\theta(p_1+p_2)+\cos\frac{p_1\theta p_3}{2}\cos\frac{p_2\theta p_4}{2}\cos k\theta(p_1+p_3)+ \\
&\qquad+\cos\frac{p_1\theta p_4}{2}\cos\frac{p_2\theta p_3}{2}\cos k\theta(p_1+p_4) \\
&\quad=\left[\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\cos k\theta(p_1+p_2)+2\text{ permutations}\right].
\end{aligned}\tag{6.40}$$

Now it can be seen that this term can be written as:

$$\begin{aligned}
&\frac{\lambda^3}{27}\left\{\int\frac{\bar{d}^4k\,\bar{d}^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)}\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\times\right. \\
&\left.\times\cos k\theta(p_1+p_2)\left[\frac{1}{(q+k+p_3)^2+m^2}+\frac{1}{(p_1+p_2-q)^2+m^2}-\frac{2}{q^2+m^2}\right]+2\text{ perm}\right\}
\end{aligned}$$

and this is obviously finite since the integration over k is regulated by the cosine factors, while for q the difference of the three propagators together with the overall $1/q^2+m^2$ is just enough to guarantee the convergence of the integral.

The next cosine combination we are going to consider is:

$$\frac{1}{4}\left[\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+k\theta(p_1+p_4)\right)+\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}+k\theta(p_1+p_3)\right)\right].\tag{6.41}$$

This can only be found in the fish diagram and in the counterterm and the total contribution is:

$$\begin{aligned}
&\frac{\lambda^3}{27}\left\{\int\frac{\bar{d}^4k\,\bar{d}^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)}\left[\frac{1}{(q+k+p_3)^2+m^2}-\frac{1}{q^2+m^2}\right]\times\right. \\
&\times\frac{1}{4}\left[\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+k\theta(p_1+p_4)\right)+\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}+k\theta(p_1+p_3)\right)\right]+ \\
&\qquad\left.+5\text{ perm}\right\}.
\end{aligned}\tag{6.42}$$

The same argument as before can be applied to show that this is finite.

Before going on to the last combination of cosines, let us note some technical fact regarding the permutations together with which various terms appear.

$$\left\{\int \frac{d^4k d^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)} \cdot \cos\frac{p_1\theta p_2}{2} \times \right.$$
$$\left. \times \left[\cos\left(\frac{p_3\theta p_4}{2} - k\theta p_3\right) + \cos\left(\frac{p_3\theta p_4}{2} + k\theta p_4\right)\right] + 5 \text{ perm}\right\} =$$
$$= \left\{\int \frac{d^4k d^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)} \cos\frac{p_1\theta p_2}{2} \times \right.$$
$$\times \left[\cos\left(\frac{p_3\theta p_4}{2} - k\theta p_3\right) + \cos\left(\frac{p_3\theta p_4}{2} + k\theta p_4\right)\right] +$$
$$+ \int \frac{d^4k d^4q}{(k^2+m^2)((p_3+p_4-k)^2+m^2)(q^2+m^2)} \cdot \cos\frac{p_3\theta p_4}{2} \times$$
$$\left. \times \left[\cos\left(\frac{p_1\theta p_2}{2} - k\theta p_1\right) + \cos\left(\frac{p_1\theta p_2}{2} + k\theta p_2\right)\right] + 2 \text{ perm}\right\}. \quad (6.43)$$

Using the conservation of momenta $(p_3+p_4-k)^2$ can be replaced by $(p_1+p_2+k)^2$. Now by changing k with $-k$ the above result can be written:

$$\left\{\int \frac{d^4k d^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)} \times \right.$$
$$\times \left[\cos\frac{p_3\theta p_4}{2}\left(\cos\left(\frac{p_1\theta p_2}{2} + k\theta p_1\right) + \cos\left(\frac{p_1\theta p_2}{2} - k\theta p_2\right)\right) + \right.$$
$$\left.\left. + \cos\frac{p_1\theta p_2}{2}\left(\cos\left(\frac{p_3\theta p_4}{2} - k\theta p_3\right) + \cos\left(\frac{p_3\theta p_4}{2} + k\theta p_4\right)\right)\right] + 2 \text{ perm}\right\}. \quad (6.44)$$

With these considerations the remaining terms in (6.39) can be written:

$$\frac{\lambda^3}{27}\left\{\int \frac{d^4k d^4q}{(k^2+m^2)((p_1+p_2-k)^2+m^2)(q^2+m^2)} \times \right.$$
$$\times \frac{1}{4}\left[\cos\frac{p_3\theta p_4}{2}\left(\cos\left(\frac{p_1\theta p_2}{2} + k\theta p_1\right) + \cos\left(\frac{p_1\theta p_2}{2} - k\theta p_2\right)\right) + \right.$$
$$\left. + \cos\frac{p_1\theta p_2}{2}\left(\cos\left(\frac{p_3\theta p_4}{2} - k\theta p_3\right) + \cos\left(\frac{p_3\theta p_4}{2} + k\theta p_4\right)\right)\right] \times$$
$$\left. \times \left[\frac{1}{(p_1+p_2-q)^2} + \frac{3}{(q+k+p_3)^2+m^2} - \frac{4}{q^2+m^2}\right] + 2 \text{ perm}\right\}. \quad (6.45)$$

As before, we notice that this term is obviously finite due to the right combination of the propagators in the last square bracket. The interesting phenomenon which can be observed in these last formulas is that the terms coming from the nonplanar parts do not really cancel, but they combine in such a way that the final result remains finite.

Summarizing, the divergences coming from the nonplanar part of diagrams (G) and (H) in eq. (6.31) are canceled against the nonplanar part of the counterterm (F). With this the proof of renormalizability of the noncommutative Φ^4 theory up to two loops is complete.

7. Conclusions and remarks

In this work we have studied the field theories written on the noncommutative Moyal plane (noncommutative field theories). These field theories are obtained by replacing the usual product of fields by the star product. First we discussed some issues of these theories at classical level, then using the usual methods we quantized the theory. We discussed both canonical and path integral methods. Because of the star product properties, the quadratic part of the action is not changed and hence only in the interaction part one can trace the noncommutativity. Extending this fact to the quantum level, we assumed that the Fock space for a commutative field theory and its noncommutative version are the same. In the path integral formulation this means that the measure for the commutative and noncommutative theories should be the same, and we support this by formulating our theory in the momentum space. We should also remind that in this work we mainly restrict ourselves to the noncommutative space; the issue of noncommutative space-time because of having some problems with unitarity and causality, seems to be more involved and subtle.

Having developed the necessary ingredients, we worked out the one and two loops two and four point functions for a noncommutative Φ^4 theory in 4 dimensions, and presented all the detailed (and maybe tedious) calculations. The important point to be noticed in the noncommutative cases is that although the counter-terms arise from the planar parts of the diagrams and hence they have the same divergence structure as the commutative case, they appear to have different numeric factors. Furthermore, these counter-terms are not θ dependent. The latter means that the $\theta \to 0$ limit is not a smooth limit, and considering the quantum (loop) corrections we are not recovering the usual commutative field theory in this limit.

The other point we should address here is that because of this IR/UV mixing it is not yet clear that the usual wilsonian renormalization and renormalization group arguments should work all the same in the noncommutative case as well.

The result which we want to emphasize on here is that the noncommutativity parameter, θ, is not receiving any loop corrections, even at two loops and we expect this result to be an exact one, i.e. θ is exact, without any quantum corrections at all loops. The other interesting question which we did not address here is the problem of gauge fields and gauge fixing, and extending the present work to gauge theories + fermions, which we hope to come back to in later works.

Acknowledgments

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A. Functional derivatives in star product formalism

In this appendix we are going to show what we mean by taking functional derivatives of terms which contain star products. First we are going to adopt the usual definition for the functional derivative, i.e.

$$S[\Phi+\delta\Phi]-S[\Phi]\equiv\int d^4x\frac{\delta S[\Phi]}{\delta\Phi(x)}\delta\Phi(x)\,. \tag{A.1}$$

Let us apply this definition to the Φ^4 theory:

$$\begin{aligned}
S_{\rm int}[\Phi+\delta\Phi]-S_{\rm int}[\Phi] = \frac{\lambda}{4!}\Big\{\int d^4x\Big[&\big((\Phi+\delta\Phi)\star\Phi\star\Phi\star\Phi\big)(x)+\\
&+\big(\Phi\star(\Phi+\delta\Phi)\star\Phi\star\Phi\big)(x)+\\
&+\big(\Phi\star\Phi\star(\Phi+\delta\Phi)\star\Phi\big)+\\
&+\big(\Phi\star\Phi\star\Phi\star(\Phi+\delta\Phi)\big)(x)\Big]-\\
&-\int d^4x\big(\Phi\star\Phi\star\Phi\star\Phi\big)(x)\Big\}\\
=\int d^4x\big(\delta\Phi\star\Phi\star\Phi\star\Phi\big)(x)&+\int d^4x\big(\Phi\star\delta\Phi\star\Phi\star\Phi\big)(x)+\\
+\int d^4x\big(\Phi\star\Phi\star\delta\Phi\star\Phi\big)(x)&+\int d^4x\big(\Phi\star\Phi\star\Phi\star\delta\Phi\big)(x).
\end{aligned} \tag{A.2}$$

Making use of the cyclic property (1.13) and of the associativity of star product (1.11) we can write:

$$\begin{aligned}
\int d^4x\frac{\delta S_{\rm int}[\Phi]}{\delta\Phi(x)}\delta\Phi(x) &= \frac{\lambda}{3!}\int d^4x\Big[\big(\Phi\star\Phi\star\Phi\big)\star\delta\Phi\Big](x)\\
&=\frac{\lambda}{3!}\int d^4x\big(\Phi\star\Phi\star\Phi\big)(x)\cdot\delta\Phi(x)
\end{aligned} \tag{A.3}$$

JHEP01(2001)025

so that we can identify

$$\frac{\delta S_{\rm int}[\Phi]}{\delta\Phi(x)} = \frac{\lambda}{3!}\big(\Phi\star\Phi\star\Phi\big)(x)\,. \tag{A.4}$$

In the path integral formalism we need to know all possible functional derivatives of our action with respect to the fields. In the following we are going to show explicitly the right way to compute these functional derivatives for the case of the scalar field theory with Φ^4 interaction. First let us note that due to the conjugation property of the star product (1.16) the r.h.s of (A.4) is real if the field Φ is real. However, the next order functional derivatives do not enjoy this property anymore, and one should make the result to be real explicitly.

$$\begin{aligned}\frac{\delta^2 S_{\rm int}}{\delta\Phi(x_1)\delta\Phi(x_2)} = -\frac{\lambda}{6}\,{\rm Re}\Bigg\{ & e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}} e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}} \times \\ & \times\Big[\delta(x_1+\xi-x_2)\Phi(x_1+\eta+\alpha)\Phi(x_1+\eta+\beta) + \\ & + \Phi(x_1+\xi)\delta(x_1+\eta+\alpha-x_2)\Phi(x_1+\eta+\beta) + \\ & + \Phi(x_1+\xi)\Phi(x_1+\eta+\alpha)\delta(x_1+\eta+\beta-x_2)\Big]\Bigg\}\,,\end{aligned} \tag{A.5}$$

$$\begin{aligned}\frac{\delta^3 S_{\rm int}}{\delta\Phi(x_1)\delta\Phi(x_2)\delta\Phi(x_3)} = -\frac{\lambda}{6}\,{\rm Re}\Bigg\{ & e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}} e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}} \times \\ & \times\Big[\delta(x_1+\xi-x_2)\delta(x_1+\eta+\alpha-x_3)\Phi(x_1+\eta+\beta) + \\ & +\delta(x_1+\xi-x_2)\Phi(x_1+\eta+\alpha)\delta(x_1+\eta+\beta-x_3) + \\ & +\delta(x_1+\xi-x_3)\delta(x_1+\eta+\alpha-x_2)\Phi(x_1+\eta+\beta) + \\ & +\Phi(x_1+\xi)\delta(x_1+\eta+\alpha-x_2)\delta(x_1+\eta+\beta-x_3) + \\ & +\delta(x_1+\xi-x_3)\Phi(x_1+\eta+\alpha)\delta(x_1+\eta+\beta-x_2) + \\ & +\Phi(x_1+\xi)\delta(x_1+\eta+\alpha-x_3)\delta(x_1+\eta+\beta-x_2)\Big]\Bigg\}\,,\end{aligned} \tag{A.6}$$

$$\begin{aligned}\frac{\delta^4 S_{\rm int}}{\delta\Phi(x_1)\delta\Phi(x_2)\delta\Phi(x_3)\Phi(x_4)} = -\frac{\lambda}{6}\,{\rm Re}\Bigg\{ & e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}} e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}} \times \\ & \times\Big[\delta(x_1+\xi-x_2)\delta(x_1+\eta+\alpha-x_3)\times \\ & \times\delta(x_1+\eta+\beta-x_4) + +\delta(x_1+\xi-x_2)\times \\ & \times\delta(x_1+\eta+\alpha-x_4)\delta(x_1+\eta+\beta-x_3) +\end{aligned} \tag{A.7}$$

$$
\begin{aligned}
&+\delta(x_1+\xi-x_3)\delta(x_1+\eta+\alpha-x_2)\times\\
&\times\delta(x_1+\eta+\beta-x_4)+\delta(x_1+\xi-x_4)\times\\
&\times\delta(x_1+\eta+\alpha-x_2)\delta(x_1+\eta+\beta-x_3)+\\
&+\delta(x_1+\xi-x_3)\delta(x_1+\eta+\alpha-x_4)\times\\
&\times\delta(x_1+\eta+\beta-x_2)+\delta(x_1+\xi-x_4)\times\\
&\times\delta(x_1+\eta+\alpha-x_3)\delta(x_1+\eta+\beta-x_2)\Big]\Bigg\}.
\end{aligned}
$$

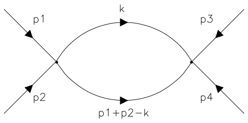
The above expression looks simpler in momentum space:

$$
\begin{aligned}
\frac{\delta^4 S_{int}}{\delta\Phi^4} &= -\frac{\lambda}{6}\,\mathrm{Re}\Bigg\{\int \bar{d}^4k_2\bar{d}^4k_3\bar{d}^4k_4 e^{ik_2(x_1-x_2)}e^{ik_3(x_1-x_3)}e^{ik_4(x_1-x_4)}e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}\times\\
&\quad\times e^{\frac{i}{2}\theta_{\rho\sigma}\partial_{\alpha_\rho}\partial_{\beta_\sigma}}\Big[e^{ik_2\xi}e^{i(k_3+k_4)\eta}e^{ik_3\alpha}e^{ik_4\beta}+e^{ik_2\xi}e^{i(k_3+k_4)\eta}e^{ik_4\alpha}e^{ik_3\beta}+\\
&\quad+e^{ik_3\xi}e^{i(k_2+k_4)\eta}e^{ik_2\alpha}e^{ik_4\beta}+e^{ik_4\xi}e^{i(k_2+k_3)\eta}e^{ik_2\alpha}e^{ik_3\beta}+\\
&\quad+e^{ik_3\xi}e^{i(k_2+k_4)\eta}e^{ik_4\alpha}e^{ik_2\beta}+e^{ik_4\xi}e^{i(k_2+k_3)\eta}e^{ik_3\alpha}e^{ik_2\beta}\Big]\Bigg\}\\
&= -\frac{\lambda}{6}\,\mathrm{Re}\Bigg\{\int \bar{d}^4k_2\bar{d}^4k_3\bar{d}^4k_4 e^{ik_2(x_1-x_2)}e^{ik_3(x_1-x_3)}e^{ik_4(x_1-x_4)}\times\\
&\quad\times\Big[e^{-\frac{i}{2}(k_2\theta(k_3+k_4))}e^{-\frac{i}{2}(k_3\theta k_4)}+e^{-\frac{i}{2}(k_2\theta(k_3+k_4))}e^{-\frac{i}{2}(k_4\theta k_3)}+\\
&\quad+e^{-\frac{i}{2}(k_3\theta(k_2+k_4))}e^{-\frac{i}{2}(k_2\theta k_4)}+e^{-\frac{i}{2}(k_3\theta(k_2+k_4))}e^{-\frac{i}{2}(k_4\theta k_2)}+\\
&\quad+e^{-\frac{i}{2}(k_4\theta(k_2+k_3))}e^{-\frac{i}{2}(k_2\theta k_3)}+e^{-\frac{i}{2}(k_4\theta(k_2+k_3))}e^{-\frac{i}{2}(k_3\theta k_2)}\Big]\Bigg\}\\
&= -\frac{\lambda}{3}\int \bar{d}^4k_1\bar{d}^4k_2\bar{d}^4k_3\bar{d}^4k_4 e^{-ik_1x_1-ik_2x_2-ik_3x_3-ik_4x_4}(2\pi)^4\delta(k_1+k_2+k_3+k_4)\times\\
&\quad\times\Big[\cos\frac{k_1\theta k_2}{2}\cos\frac{k_3\theta k_4}{2}+\cos\frac{k_1\theta k_3}{2}\cos\frac{k_2\theta k_4}{2}+\cos\frac{k_1\theta k_4}{2}\cos\frac{k_2\theta k_3}{2}\Big].
\end{aligned}
\tag{A.8}
$$

B. Phase factors associated with various diagrams

In this section we are going to compute explicitly the phase factors associated with the diagrams we encounter in the two loop expansion of the four point function (6.31)

Let us first introduce a notation for the factors associated with the one loop diagram:

$\frac{1}{2}$ 

$\propto \mathcal{P}(p_1\ldots p_4,k,\theta)$, where (B.1)

$$
\begin{aligned}
\mathcal{P}(p_1 \dots p_4, k, \theta) = {}& \frac{\lambda^2}{9} \cos\frac{p_1\theta p_2}{2} \cos\frac{p_3\theta p_4}{2} \left[1 + \cos k\theta(p_1 + p_2) \right] + \\
& + \frac{\lambda^2}{18} \cos\frac{p_3\theta p_4}{2} \left[\cos\left(\frac{p_1\theta p_2}{2} + k\theta p_1 \right) + \cos\left(\frac{p_1\theta p_2}{2} - k\theta p_2 \right) \right] + \\
& + \frac{\lambda^2}{18} \cos\frac{p_1\theta p_2}{2} \left[\cos\left(\frac{p_3\theta p_4}{2} - k\theta p_3 \right) + \cos\left(\frac{p_3\theta p_4}{2} + k\theta p_4 \right) \right] + \\
& + \frac{\lambda^2}{36} \left[\cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2} + k\theta(p_1 + p_3) \right) + \right. \\
& \qquad \left. + \cos\left(\frac{p_1\theta p_2 + p_3\theta p_4}{2} + k\theta(p_1 + p_4) \right) \right] .
\end{aligned} \tag{B.2}
$$

If we want to find the planar part of this graph we should just drop the terms which involve internal momenta and taking into account the two permutations together with which this diagram appears in the loop expansion we conclude:

$$
Planar[\mathcal{P}(p_1 \dots p_4, k, \theta)] = \frac{\lambda^2}{9} \left(\cos\frac{p_1\theta p_2}{2} \cos\frac{p_3\theta p_4}{2} + 2 \text{ perm} \right) , \tag{B.3}
$$

so that the momentum dependent factor is just the same as the usual factor we added for the noncommutative vertex (3.12). We shall now compute the phase factors associated to each diagram in the expansion of $\Gamma^{(4)}$ in (6.31) leaving apart for the moment the overall factor 1/27.

$$
\begin{aligned}
(G) \propto {}& \left[\cos\frac{(k-q)\theta(p_1+p_2)}{2} + \cos\left(\frac{(k+q)\theta(p_1+p_2)}{2} - k\theta q \right) + \cos\frac{(k+q)\theta(p_1+p_2)}{2} \right] \times \\
& \times \left[\cos\frac{p_1\theta p_2 + k\theta(p_1+p_2)}{2} + \cos\frac{p_1\theta p_2 - k\theta(p_1+p_2)}{2} + \cos\frac{p_1\theta p_2 + k\theta(p_1-p_2)}{2} \right] \times \\
& \times \left[\cos\frac{p_3\theta p_4 + q\theta(p_3+p_4)}{2} + \cos\frac{p_3\theta p_4 - q\theta(p_3+p_4)}{2} + \cos\frac{p_3\theta p_4 - q\theta(p_3-p_4)}{2} \right] .
\end{aligned} \tag{B.4}
$$

At two-loops we have two independent internal momenta to integrate over so, as explained in section 6.3 all the terms which contain cosine factors and involve both of these momenta will remain finite after integrations. The second term in the first factor from eq. (B.4) contains a $k\theta q$ in the argument of the cosine which cannot be found anywhere else. So after expanding and transforming the cosine products into sums of cosines this term cannot disappear. In what follows we consider only terms from which either k or q (or both) disappear. These terms come from:

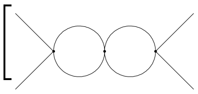
$$
\frac{1}{2} \left[\cos\frac{p_3\theta p_4 + q\theta(p_3+p_4)}{2} + \cos\frac{p_3\theta p_4 - q\theta(p_3+p_4)}{2} + \cos\frac{p_3\theta p_4 + q\theta(p_3-p_4)}{2} \right] \times
$$

$$
\begin{aligned}
&\times\Bigg[\cos\left(\frac{p_1\theta p_2 - q\theta(p_1+p_2)}{2} + k\theta(p_1+p_2)\right) + \cos\left(\frac{p_1\theta p_2 + q\theta(p_1+p_2)}{2}\right) + \\
&+\cos\left(\frac{p_1\theta p_2 + q\theta(p_1+p_2)}{2} - k\theta(p_1+p_2)\right) + \cos\left(\frac{p_1\theta p_2 - q\theta(p_1+p_2)}{2}\right) + \\
&+\cos\left(\frac{p_1\theta p_2 + q\theta(p_1+p_2)}{2} + k\theta(p_1+p_2)\right) + \cos\left(\frac{p_1\theta p_2 - q\theta(p_1+p_2)}{2}\right) + \\
&+\cos\left(\frac{p_1\theta p_2 - q\theta(p_1+p_2)}{2} - k\theta(p_1+p_2)\right) + \cos\left(\frac{p_1\theta p_2 + q\theta(p_1+p_2)}{2}\right) + \\
&+\cos\left(\frac{p_1\theta p_2 - q\theta(p_1+p_2)}{2} + k\theta p_1\right) + \cos\left(\frac{p_1\theta p_2 + q\theta(p_1+p_2)}{2} - k\theta p_2\right) + \\
&+\cos\left(\frac{p_1\theta p_2 + q\theta(p_1+p_2)}{2} + k\theta p_1\right) + \cos\left(\frac{p_1\theta p_2 - q\theta(p_1+p_2)}{2} - k\theta p_2\right)\Bigg]. \qquad \text{(B.5)}
\end{aligned}
$$

The planar part. Using overall momentum conservation we can extract the planar part:

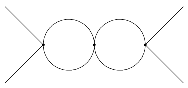
$$
\begin{aligned}
\text{Planar}[(G)] &= 2\cdot\frac{1}{2}\left[\cos\frac{p_3\theta p_4 + q\theta(p_3+p_4)}{2} + \cos\frac{p_3\theta p_4 - q\theta(p_3+p_4)}{2}\right]\times \\
&\quad\times\left[\cos\frac{p_1\theta p_2 + q\theta(p_1+p_2)}{2} + \cos\frac{p_1\theta p_2 - q\theta(p_1+p_2)}{2}\right]\Bigg|_{\text{Planar}} = \\
&= \cos\frac{p_1\theta p_2 + p_3\theta p_4}{2} + \cos\frac{p_1\theta p_2 - p_3\theta p_4}{2} \\
&= 2\cdot\cos\frac{p_1\theta p_2}{2}\cdot\cos\frac{p_3\theta p_4}{2}. \qquad \text{(B.6)}
\end{aligned}
$$

The “2 permutations” take care of the other combinations of external momenta, and so

$\frac{1}{4}\Bigg[$  $+2\text{permutations}\Bigg]_{\text{planar}}$

$$
= \frac{1}{2\cdot 27}\left[\cos\frac{p_1\theta p_2}{2}\cdot\cos\frac{p_3\theta p_4}{2} + \cos\frac{p_1\theta p_3}{2}\times\cos\frac{p_2\theta p_4}{2} + \cos\frac{p_1\theta p_4}{2}\cdot\cos\frac{p_2\theta p_3}{2}\right]. \qquad \text{(B.7)}
$$

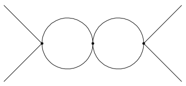
Nonplanar q-independent terms.

$\frac{1}{4}$  $\Big|_{\text{NP I}}$

$$
\begin{aligned}
= \frac{1}{16}\Bigg[&2\cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2} + k\theta(p_1+p_2)\right) + \qquad \text{(B.8)}\\
&+2\cos\left(\frac{p_1\theta p_2 + p_3\theta p_4}{2} + k\theta(p_1+p_2)\right) +
\end{aligned}
$$

$$
\begin{aligned}
&+2\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}-k\theta(p_1+p_2)\right)+\\
&+2\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}-k\theta(p_1+p_2)\right)+\\
&+2\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}+k\theta p_1\right)+\\
&+2\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+k\theta p_1\right)+\\
&+2\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}-k\theta p_2\right)+\\
&+2\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}-k\theta p_2\right)\Bigg]\\
=\;&\frac{1}{2}\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\cos k\theta(p_1+p_2)+\\
&+\frac{1}{4}\cos\frac{p_3\theta p_4}{2}\left[\cos\left(\frac{p_1\theta p_2}{2}+k\theta p_1\right)+\cos\left(\frac{p_1\theta p_2}{2}-k\theta p_2\right)\right].
\end{aligned}
$$

Nonplanar k-independent terms. The diagram is perfectly symmetric in k and $-q$ so we can just replace k by $-q$ to get:

$\frac{1}{4}$  $\Big|_{\text{NP II}}$

$$
\begin{aligned}
&= \frac{1}{2}\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\cos q\theta(p_1+p_2)+\frac{1}{4}\cos\frac{p_1\theta p_2}{2}\times\\
&\times\left[\cos\left(\frac{p_3\theta p_4}{2}-q\theta p_3\right)+\cos\left(\frac{p_3\theta p_4}{2}+q\theta p_4\right)\right]. \qquad \text{(B.9)}
\end{aligned}
$$

Let us now consider the next term in eq. (6.31). The phase factors associated with the vertices are:

$$
\begin{aligned}
(H)\propto&\left[\cos\frac{p_1\theta p_2+k\theta(p_1+p_2)}{2}+\cos\frac{p_1\theta p_2-k\theta(p_1+p_2)}{2}+\cos\frac{p_1\theta p_2+k\theta(p_1-p_2)}{2}\right]\times\\
&\times\left[\cos\frac{p_3\theta k-q\theta p_3-q\theta k}{2}+\cos\frac{p_3\theta k+q\theta p_3+q\theta k}{2}+\cos\frac{p_3\theta k-q\theta p_3+q\theta k}{2}\right]\times\\
&\times\Bigg[\cos\frac{p_3\theta p_4+k\theta p_4+q\theta p_3+q\theta k}{2}+\cos\frac{p_3\theta p_4+k\theta p_4-q\theta p_3-q\theta k}{2}+\\
&+\cos\left(\frac{p_3\theta p_4+k\theta p_4+q\theta p_3+q\theta k}{2}+q\theta p_4\right)\Bigg]. \qquad \text{(B.10)}
\end{aligned}
$$

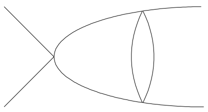
As before the terms containing simultaneously k and q in the argument of cosine factor give no contribution to the divergent part. This means that in the product of the last two terms we only have to consider those combinations of cosines which do not lead to factors of $k\theta q$ in the argument. It is easy to see that these terms come from:

$$
\begin{aligned}
&\frac{1}{2}\Bigg[2\cos\frac{p_3\theta p_4 - k\theta(p_3-p_4)}{2} + 2\cos\frac{p_3\theta p_4 + k\theta(p_3+p_4)}{2} + \\
&\quad + \cos\left(\frac{p_3\theta p_4 - k\theta(p_3-p_4)}{2} + q\theta p_4\right) + \cos\left(\frac{p_3\theta p_4 + k\theta(p_3+p_4)}{2} + q\theta p_4\right) + \\
&\quad + \cos\left(\frac{p_3\theta p_4 + k\theta(p_3+p_4)}{2} - q\theta p_3\right) + \cos\left(\frac{p_3\theta p_4 + k\theta(p_4-p_3)}{2} + q\theta p_3\right) + \\
&\quad + \cos\left(\frac{p_3\theta p_4 + k\theta(p_3+p_4)}{2} + q\theta(p_3+p_4)\right)\Bigg] \times \\
&\times\Bigg[\cos\frac{p_1\theta p_2 + k\theta(p_1+p_2)}{2} + \cos\frac{p_1\theta p_2 - k\theta(p_1+p_2)}{2} + \\
&\qquad\qquad + \cos\frac{p_1\theta p_2 + k\theta(p_1-p_2)}{2}\Bigg]. \qquad \text{(B.11)}
\end{aligned}
$$

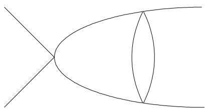
The planar part. Proceeding in the same way as before we can write the planar part:

$$
\begin{aligned}
&\cos\frac{p_3\theta p_4 + k\theta(p_3+p_4)}{2}\cdot\left[\cos\frac{p_1\theta p_2 + k\theta(p_1+p_2)}{2} + \cos\frac{p_1\theta p_2 - k\theta(p_1+p_2)}{2}\right]\Bigg|_{\text{Planar}} = \\
&= \frac{1}{2}\left[\cos\frac{p_1\theta p_2 + p_3\theta p_4}{2} + \cos\frac{p_1\theta p_2 - p_3\theta p_4}{2}\right]. \qquad \text{(B.12)}
\end{aligned}
$$

Taking into account all the coefficients (also the 1/27) we obtain:

$$\frac{1}{2}$$  $$\Bigg|_{\text{Planar}} \propto \frac{1}{2\cdot 27}\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}. \qquad \text{(B.13)}$$

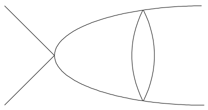
Nonplanar q-independent terms.

 $$\Bigg|_{\text{NP I}} \propto$$

$$
\begin{aligned}
&\propto\left[\cos\frac{p_1\theta p_2 + k\theta(p_1+p_2)}{2} + \cos\frac{p_1\theta p_2 - k\theta(p_1+p_2)}{2} + \cos\frac{p_1\theta p_2 + k\theta(p_1-p_2)}{2}\right]\times \\
&\quad\times\left[\cos\frac{p_3\theta p_4 + k\theta(p_4-p_3)}{2} + \cos\frac{p_3\theta p_4 + k\theta(p_3+p_4)}{2}\right]\Bigg|_{\text{nonplanar}} = \\
&= \frac{1}{2}\Bigg[\cos\left(\frac{p_1\theta p_2 + p_3\theta p_4}{2} - k\theta p_3\right) + \cos\left(\frac{p_1\theta p_2 - p_3\theta p_4}{2} - k\theta p_4\right) +
\end{aligned}
$$

$$
\begin{aligned}
&+\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+k\theta p_4\right)+\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}+k\theta p_3\right)+\\
&+\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+k\theta(p_1+p_4)\right)+\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}+k\theta(p_1+p_3)\right)+\\
&+\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}+k\theta(p_1+p_2)\right)+\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}-k\theta(p_1+p_2)\right)+\\
&+\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}-k\theta p_2\right)+\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}+k\theta p_1\right)\Bigg]\,. \qquad \text{(B.14)}
\end{aligned}
$$

Nonplanar k-independent terms.



$$\Big|_{\mathrm{NP\ II}} \propto$$

$$
\begin{aligned}
\propto \frac{1}{4}\Bigg[&\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+q\theta p_4\right)+\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}-q\theta p_4\right)\\
&+\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+q\theta p_3\right)+\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}-q\theta p_3\right)\\
&+\cos\left(\frac{p_1\theta p_2+p_3\theta p_4}{2}+q\theta(p_3+p_4)\right)+\cos\left(\frac{p_1\theta p_2-p_3\theta p_4}{2}-q\theta(p_3+p_4)\right)\Bigg]\,.
\end{aligned}
\qquad \text{(B.15)}
$$

Summarizing, the main result we found in this appendix is that the planar parts of the loop diagrams in (6.31) come in with the same momentum dependent phase factor, which is exactly (3.12). In this way we motivated the claim that we can renormalize the planar parts as in the commutative case.

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